Phys333 HW5 Solutions

#1 GMC dust extinction & reddening

2. *Quick Question 19-2: GMC dust extinction and reddening*

   a. For a GMC with molecular Hydrogen density \( n = 100 \text{ cm}^{-3} \), compute the associated mass density \( \rho \).

   b. For UV light with \( \lambda = 100 \text{ nm} \) and dust of comparable size \( a = 0.1 \mu\text{m} \) with the solar-abundance mass fraction \( X_d = 2 \times 10^{-3} \), use the geometric cross section opacity derived in DocOnotes §19.3 to compute the mean free path \( \ell \) (in pc) for this GMC.

   c. For a GMC of diameter \( D = 30 \text{ pc} \), compute the number of magnitudes of extinction, \( A_{UV} \), for this UV light.

   d. Assuming reddening exponent \( \beta = 1 \), now compute the extinction \( A_V \) for visible light with \( \lambda = 500 \text{ nm} \), and the extinction \( A_{NIR} \) for near IR light with \( \lambda = 2 \mu\text{m} \).

   e. Compared to what would be observed if there were no absorption from ISM dust, by what factor \( f \) is observed flux reduced by dust absorption in the Visible and the in NIR?
2. \( \eta = 100 \text{ cm}^3 \text{ mol}^{-1} \text{ K} \text{ g}^{-1} \)

a. \( \rho = \eta n = 3.3 \text{ g cm}^{-3} \)

b. \( K_d = 150 \text{ cm}^2 \text{ s}^{-1} \mu \text{ m}^{-3} \) \( p_c = 3.16 \text{ cm} \)

\[ l = \frac{\lambda}{K_d} = \frac{\lambda}{1.52 \cdot 3.3 \cdot 22.3} = 6.6 \text{ pC} \]

c. \( d = 30 \text{ pC} \) \( \tau = \frac{d}{l} = \frac{30}{6.6} = 4.5 \)

\[ \Lambda_{nv} = 1.09 \tau = 4.9 \]

d. \( \Lambda_d (\lambda) = 150 \text{ cm}^2 \text{ s}^{-1} \mu \text{ m}^{-3} \)

\( \lambda_v = 500 \text{ nm} = 0.5 \mu \text{ m} \) \( \Rightarrow K_d = \frac{150}{5} = 30 \text{ cm}^2 \text{ s}^{-1} \mu \text{ m}^{-3} \)

\[ \Lambda_v = \Lambda_{uv} / 5 = 1 \]

\( \lambda_{IR} = 2 \mu \text{ m} \) \( \Rightarrow K_d = \frac{150}{20} = 7.5 \text{ cm}^2 \text{ s}^{-1} \mu \text{ m}^{-3} \)

\[ \Lambda_{IR} = \frac{\Lambda_{uv}}{20} = 0.25 \]

e. \( f = e^{-x} \) \( \tau_v = 0.9 \) \( f_v = e^{-0.9} = 0.4 \)

\( \tau_w = 0.72 \) \( f_{IR} = e^{-0.22} = 0.8 \)
3. Exercise 20-2: Flattened Salpeter IMF

a. For the simple flattened Salpeter IMF, with $\alpha = 2.35$ for $m > 1$ and $\alpha = 0$ for $m < 1$, integrate Eq. 20.8 over all masses to obtain an expression for the normalization $K$ in terms of the total number of stars $N_{tot}$.

b. Now use this to obtain an expression for the fraction of stars, $N(m > m_0)/N_{tot}$, with mass greater than some mass lower limit $m_0$ (assuming $m_0 \geq 1$). In particular, what fraction of stars have $m > 1$?

c. For $m_0 = 100$, how many total stars must a cluster have for there to be at least one star with $m \geq m_0$? How about for $m_0 = 300$? What does this imply for observational efforts to determine whether there is an upper mass cutoff to the IMF?
#3 Isobaric ISM

4. a. Assuming an isobaric ISM with the canonical pressure \( P/k = nT = 10^3 \text{ K cm}^{-3} \), use DocOnotes eqns. (20.4) and (20.5) to derive expressions for the Jean’s radius \( R_J \) (in pc) and Jean’s mass \( M_J \) (in \( M_\odot \)) as a function of temperature \( T \) (in K).

b. Now derive analogous expressions for \( R_J \) and \( M_J \) as functions of number density \( n \) (in \( \text{cm}^{-3} \)).

c. Assuming the molecular weight \( \mu = 2m_p \) for pure molecular hydrogen, what would be the density \( n \) (in \( \text{cm}^{-3} \)) of a Jean’s cloud with this pressure and a mass \( M = 1000M_\odot \)?

d. What would be the associated Jean’s radius \( R_J \) (in pc)?

e. What angle (in arcsec) would the cloud diameter subtend at a distance of 1 kpc from the Earth?

f. Finally, what would be the free-fall time \( t_{ff} \) (in years)?
2a. (20.4) \( R_J = 9.6 \frac{(I_m^2 m_p)^{1/2}}{\mu} \text{ pc} \)

(20.15) \( M_J = 9.2 \frac{I^{3/2}}{\nu^2} \left( \frac{m_p}{\mu} \right)^2 M_\odot \)

\( n_T = 10^3 \)

\[ R_J = \frac{9.6 \times 10^{3/2}}{10^{3/2}} \frac{m_p}{\mu} = 0.3 \text{ pc} \]

\[ M_J = \frac{9.2 \times 10^{9/2}}{10^{3/2}} \left( \frac{m_p}{\mu} \right)^2 = 3 M_\odot \]

b. \( R_S = 9.6 \times 10^{3/2} \frac{m_p}{\mu} = \frac{300 \text{ pc}}{\mu} \)

\[ M_J = \frac{9.2 \times 10^{9/2}}{3.6} \left( \frac{m_p}{\mu} \right)^2 = 3 \frac{M_\odot}{\mu} \]

C. \( 1000 M_\odot = 3 \frac{M_\odot}{\mu} \Rightarrow \mu = \frac{\sqrt{30}}{2} 10 = 27 \text{ cm}^{-3} \)

d. \( R_J = \frac{300}{2} = 5.5 \text{ pc} \)

e. \( \alpha = \frac{2R_J}{d} = \frac{11}{13} 2 \Rightarrow 2 \beta = 2.73 = 720 \text{ arcsecs} \)

\( T_{eff} = \frac{51 m_H}{\sqrt{\mu}} \sqrt{\frac{2 m_p}{\mu}} = \frac{51}{\sqrt{2}} = 10 \text{ Myr} \)
5. *HII Regions*

Consider a star with temperature 30,000 K (about $5 \times T_\odot$) and radius $R = 40 \, R_\odot$.

a. Compute the total luminosity $L$ in $L_\odot$.

b. For the Hydrogen ionization energy $E = h\nu_o = 13.6$ eV, compute the ratio $h\nu_o/kT$.

c. Using the fact that $h\nu_o/kT \gg 1$, one can show that the dimensionless ratio, $b \equiv \dot{N}_{uw}h\nu_o/L \approx 0.17$, where $\dot{N}_{uw}$ is the number of H-ionization UV photons (defined in DocOnotes eqn. 19.4). Use this to compute a numerical value for $\dot{N}_{uw}$ (in photons/sec).

d. Assuming an interstellar Hydrogen density $n = 8000 \, \text{cm}^{-3}$, compute the associated equilibrium (Stromgren) radius $R_s$ of the resulting HII region, in pc.

e. If such an HII region has an apparent angular diameter of 1 arcmin, use this and the answer to part d to determine the distance $D$ (in kpc).

f. *Bonus challenge (5-points Extra Credit).* Using the approximation $h\nu_o/kT \gg 1$ and the definitions of the Planck function and $\dot{N}_{uw}$ from the DocOnotes, compute from first principals the value of the ratio $b$ in part c, confirming the value given there.
4. HII regions

a. \( \frac{L}{L_\odot} = \frac{T_0}{T_\odot} = \frac{5.4 \times 10^4}{10^4} = \boxed{5.4} \)

b. \( \frac{E}{K} = \frac{13.6 \times 10^{-12}}{3.0 \times 10^{-12}} \frac{eV}{eV} = \boxed{5.2} \)

c. \( b \Rightarrow N_0 \frac{E}{Z} = 0.12 \Rightarrow N_0 = 0.12 \frac{1}{E} = 0.17 \Rightarrow 1.6 \Rightarrow 0.9 \Rightarrow \boxed{0.49} \)

d. \( \rho \leq 6 \Rightarrow \rho < \left[ \frac{\sqrt{N_{50}}}{N_{2}} \right]^{1/3} = 6 \left[ \frac{0.3}{8.6} \right]^{1/3} = 0.2 \Rightarrow \boxed{0.2 \rho_c} \)

e. \( d = 7 \Rightarrow \rho < \rho_c \)

\( \frac{d}{\rho_c} = \frac{5/60}{0.225} = 0.225 \Rightarrow 1300 \Rightarrow \boxed{1300 \, \rho_c} \)
#5 Radial velocity of habitable planets

5. Radial velocity detection of habitable planets

a. From eqn. (24.1) compute the amplitude of the Sun’s wobble speed (in m/s) due to Jupiter’s orbit. (You’ll need to look up Jupiter’s orbital period and its mass ratio to the Sun.)

b. Ignoring Jupiter and other planets, similarly compute the amplitude of the Sun’s wobble speed (in m/s) due to the Earth’s orbit.

c. The smallest wobble speed that can be currently measured in a star’s spectrum is about 1 m/s. What does this imply about our ability to detect analogs of Jupiter and Earth around stars with mass comparable to our Sun.

d. Next combine eqns. (24.1) and (24.3) to derive an expression for the wobble speed of a star with mass $M_*$ and temperature $T_*$ due to a habitable zone planet of mass $m_p$ and orbital period $P_e$.

e. For a star with a mass and temperature that are both 1/2 that of the Sun, estimate the smallest mass planet $m_p$ (in units of Earth’s mass $m_e$) that could be detected in this star’s habitable zone.

f. How far (in au) would such a planet be from its star?

a.

$PJ = 11.9 \text{ yr}$; (* Jupiter Period in years *)

$MJbMs = 9.4 \times 10^{-4}$; (* Mass Jupiter to Mass Sun *)

$V_{sun} = 30. \times 10^3 \text{ m/ s \ (PJ/ yr)}^{-1/3} MJbMs$; (* Eqn. 24.1 converted to m/s *)

Framed["$V_{sun} =$ NumberForm[$V_{sun}$, 3]]

\[ V_{sun} = \frac{12.4 \text{ m}}{\text{s}} \]

b.

$MebMs = 3 \times 10^{-6}$; (* Mass Earth to Mass Sun *)

$V_{sun} = 30. \times 10^5 \text{ cm/ s \ MebMs}$; (* Eqn. 24.1 converted to cm/s *)

Framed["$V_{sun} =$ NumberForm[$V_{sun}$, 3]]

\[ V_{sun} = \frac{9. \text{ cm}}{\text{s}} \]
c. At 1 m/s precision, we can readily detect Jupiters by wobble method, but not Earths.

\[ V_x = 3.0 \text{ km/s} \left( \frac{M_x}{M_\oplus} \right)^{1/3} \frac{m_p}{M_x} \]

\[ P_e = \left( \frac{T_x}{10} \right)^3 \left( \frac{M_\oplus}{M_x} \right)^{1/2} \]

\[ V_x = 3.0 \frac{\text{km}}{\text{s}} \frac{T_x}{10} \frac{M_\oplus}{M_x} \frac{m_p}{M_\oplus} \]

\[ P_e = \left( \frac{T_x}{10} \right)^3 \left( \frac{M_\oplus}{M_x} \right)^{1/2} \]

\[ m_p = \frac{100}{14.52} = 3.9 \]

\[ m_p = 3.9 \text{ } m_\oplus \]

\[ P_e = \left( \frac{T_x}{10} \right)^3 \left( \frac{M_\oplus}{M_x} \right)^{1/2} = 4 \sqrt{2} \]

\[ k_\text{III} = \frac{M_x}{M_\oplus} \left( \frac{x}{a} \right)^3 \left( \frac{a}{1.0} \right)^2 \]

\[ a = \left( \frac{T_x}{M_\oplus} \right)^{3/2} \left( \frac{P}{M_\oplus} \right)^{1/3} = \left( \frac{1}{2} \right)^{1/3} \left( 4 \sqrt{2} \right)^{2/3} = 16^{1/3} \]

\[ a = 2.52 \text{ au} \]