Polytropes

Although the equation of hydrostatic equilibrium may be integrated once to
yield the Virial theorem, it’s not generally possible to obtain an actual solution
to the equation without introducing further equations describing the tempera-
ture distribution, $T(r)$ or $T(m)$, throughout the star. Typically, this distribution
is controlled by the processes releasing energy inside the star (e.g., nuclear fu-
sion), and the processes responsible for transporting this energy (e.g., radiative
diffusion, convection).

If we can somehow eliminate the temperature $T$ from our equations, how-
ever, we don’t need to worry about these complex processes. There are two
circumstances where this can be done:

1. Assume that the gas obeys an equation of state that does not explicitly
   involve the temperature. For instance, a fully-degenerate electron gas has
   an equation of state $P \sim (\rho/\mu_e)^{4/3}$ if it is relativistic\(^1\), or $P \sim (\rho/\mu_e)^{5/3}$
   if it is non-relativistic, with no dependence on $T$.

2. Assume that the gas obeys a normal (e.g., ideal) equation of state,
   $P = \rho R_g T/\mu$, but the temperature throughout the star obeys a specific relation
   $T \sim \rho^{1/n}$ for some index $n$. This relation represents some kind of
   approximation to the star’s true temperature structure; for instance, if we
   take $n = \infty$, then we have assumed an isothermal temperature structure,
   with $T \sim \text{const.}$ — which may or may not be accurate!\(^1\)

Both of these scenarios can be modeled by polytropes, simple stellar models in
which there is a pressure-density relation of the form

$$P = K \rho^{\frac{1}{n+1}}. \quad (1)$$

Here, $K$ is a constant that depends on physical (quantum-mechanical) constants
in case (i) above, and on the overall temperature scaling in the star in case (ii).
The polytropic index $n$ (not to be confused with the particle number density) is
usually assumed to be constant throughout the star, although there exist some
specialized models (not considered here) that allow it to vary.

To find the structure of a polytrope, we write the equation of hydrostatic
equilibrium in terms of the potential gradient

$$\frac{dP}{dr} = - \frac{Gm}{r^2} \rho = - \frac{d\Phi}{dr} \rho. \quad (2)$$

\(^1\)Here, $\mu_e$ is the mean molecular weight per electron, which is 1 for fully-ionized hydrogen,
and 2 for fully-ionized helium.
Substituting in the polytropic equation (1), this becomes

$$K(1 + 1/n)\rho^{1/n-1}d\rho = -\frac{d\Phi}{dr}. \tag{3}$$

Multiplying through by $dr$ and integrating, we have

$$\int K(1 + 1/n)\rho^{1/n-1}d\rho = -\int d\Phi. \tag{4}$$

This can be calculated trivially to find

$$K(n + 1)\rho^{1/n} = -\Phi, \tag{5}$$

where we adopt the convention that $\Phi = 0$ at the stellar surface where $\rho = 0$. Note that this is different to our usual convention of setting $\Phi = 0$ at infinity.

We combine this last equation with Poisson’s equation in spherical symmetry,

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho; \tag{6}$$

eliminating the density, we have

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \left( \frac{-\Phi}{K(n + 1)} \right)^n. \tag{7}$$

To simplify things, let’s introduce a new dependent variable,

$$w = \frac{\Phi}{\Phi_c} = \left( \frac{\rho}{\rho_c} \right)^{1/n}, \tag{8}$$

where $\Phi_c$ and $\rho_c$ are the central potential and density, respectively. Likewise, we introduce a new independent variable

$$z = Ar \tag{9}$$

where the quantity $A$, with dimensions of inverse length, obeys

$$A^2 = \frac{4\pi G}{K^n(n + 1)^n} (-\Phi_c)^{n-1} = \frac{4\pi G}{K(n + 1)} \rho_c^{1-1/n}. \tag{10}$$

These two substitutions lead to a nice, clean-looking differential equation,

$$\frac{1}{z^2} \frac{d}{dz} \left( z^2 \frac{dw}{dz} \right) = -w^n. \tag{11}$$

This last equation, a second-order non-linear differential equation, has a name: it is the Lane-Emden equation, named after the 19th-C astrophysicists Jonathan Homer Lane (USA) and Robert Emden (Switzerland). From 1870 to 1910, the Lane-Emden equation — and the polytropic models that are its solution — represented the state-of-the-art in stellar astrophysics.
To solve the Lane-Emden equation, we first need to think about boundary conditions. At the center \( z \to 0 \), \( \Phi \to \Phi_c \) and \( \rho \to \rho_c \), and thus \( w \to 1 \). In addition, we know that the gravity (a.k.a. the potential gradient \( d\Phi/dr \)) must vanish at the center, so that \( w'(z) \equiv dw/dz \to 0 \). Finally, at the surface \( z = z_s \) the density must vanish, so that \( w \to 0 \). Thus, to summarize, we apply the boundary conditions

\[
w(0) = 1, \quad w'(0) = 0, \quad w(z_s) = 0. \quad (12)
\]

Solving the Lane-Emden equation (11) can only be done analytically for \( n = 0, n = 1 \) and \( n = 5 \). The \( n = 0 \) solution

\[
w(z) = 1 - \frac{1}{6} z^2 \quad (13)
\]
corresponds to a constant-density model, and has its surface at \( z_s = \sqrt{6} \approx 2.45 \). The \( n = 1 \) solution

\[
w(z) = \frac{\sin z}{z} \quad (14)
\]
has its surface at \( z_s = \pi \). The \( z = 5 \) solution

\[
w(z) = \frac{1}{\sqrt{1 + z^2/3}} \quad (15)
\]
does not reach the zero-density surface \( (w(z) = 0) \) until \( z \to \infty \); this is a general property of all polytropes with \( n \geq 5 \), including the isothermal polytrope that has \( n = \infty \).

Once we have solved the Lane-Emden equation, we can use various identities to calculate the fundamental parameters of the resulting polytropic model. The simplest of these is the relationship between the stellar radius \( R \) and the outer boundary location \( z_s \); from eqn. (9), we have

\[
R = z_s A. \quad (16)
\]
The corresponding stellar mass is given by integrating over the density,

\[
M = \int_0^R 4\pi r^2 \rho dr. \quad (17)
\]
In terms of \( w \) and \( z \), this becomes

\[
M = \frac{4\pi \rho_c}{A^4} \int_0^{z_s} z^2 w^n dz. \quad (18)
\]
The Lane-Emden equation itself can be used to rewrite this as

\[
M = -\frac{4\pi \rho_c R^3}{z_s^3} \int_0^{z_s} \frac{d}{dz} \left( z^2 \frac{dw}{dz} \right) dz, \quad (19)
\]
which is trivially integrated to find

$$M = 4\pi \rho_c R^3 \left( \frac{1}{z} \frac{dw}{dz} \right)_{z=z_s}. \quad (20)$$

Since the mean density is given by $\bar{\rho} = 3M/(4\pi R^3)$, we find that the ratio of mean-to-central density is given by

$$\frac{\bar{\rho}}{\rho_c} = \left( -\frac{3}{z} \frac{dw}{dz} \right)_{z=z_s}. \quad (21)$$

This latter expression depends only on the solution to the Lane-Emden equation, and therefore only on the polytropic index $n$. In fact, there is a monotonic relationship between density ratio and $n$; stars with larger $n$ are more centrally-condensed.

With the foregoing relations, we can begin to construct stellar models of specified mass and radius. Consider the Sun, which we shall try to approximate using an $n = 3$ polytrope. This polytrope has $z_s = 6.90$ and $z_s^2 w'(z_s) = -2.02$ (see Kippenhahn & Weigert, table 19.1). With $M = 1 \, M_\odot$ and $R = 1 \, M_\odot$, we find from equation (20) a central density $\rho_c \approx 77 \, \text{gcm}^{-3}$. From equation (16), the constant $A$ will equal $9.9 \times 10^{-11} \, \text{cm}^{-1}$, which by equation (10) implies that $K = 3.85 \times 10^{14}$. Assuming that the material in the Sun behaves as an ideal gas with mean molecular weight $\mu = 0.62$, we therefore find a central temperature $T_c \approx 1.2 \times 10^7 \, \text{K}$, which compares very favorably with the EZ Web value $T_c = 1.6 \times 10^7 \, \text{K}$. The agreement between the pressures is likewise good: $P_c \approx 1.2 \times 10^{17} \, \text{dyn cm}^{-2}$ from the $n = 3$ polytrope, compared to $P_c = 2.5 \times 10^{17} \, \text{dyn cm}^{-2}$ from EZ Web.

This application to the Sun is an illustration of circumstance (ii), where we assume that the material is an ideal gas, with a temperature distribution throughout the star obeying a relation given by an $n = 3$ polytrope: $T \sim \rho^{1/3}$. Let’s now look at circumstance (i), where the polytropic pressure-density relation (1) is the equation of state. Consider for instance a star in which the material is a non-relativistic degenerate electron gas, obeying the equation of state

$$P = 1.0036 \times 10^{13} \left( \frac{\rho}{\mu_e} \right)^{5/3}. \quad (22)$$

Comparing this against the polytropic relation (1), we see that this star can be modeled by a polytrope having $n = 3/2$ and $K = 1.0036 \times 10^{13} \mu_e^{-5/3}$. Unlike the example above with the Sun, $K$ is not a free parameter to be determined — it is fixed by quantum mechanics (see Kippenhahn & Weigert, their eqn. 15.23) and the composition of the gas. Since we’ve eliminated one of our degrees of freedom, we can expect that we will no longer be at liberty to chose $M$ and $R$; these two variables are no longer independent.

To check this, we start with eqn. (10), and eliminate $A$ using the radius.
relation (16) and \( \rho_c \) using the mass relation (20). We thus find that

\[
\frac{z_s^2}{R^2} = \frac{4\pi G}{K(n+1)} \left[ \frac{M}{4\pi R^3} \left( \frac{1}{z_s} \frac{dw}{dz} \right)_{z=z_s} \right]^{1-1/n}.
\] (23)

Rearranging, and remembering that \( K \) is now a constant, we can see that there is a mass-radius relation of the form

\[
R \sim M^{n-1/n}.
\] (24)

For our degenerate star with \( n = 3/2 \), this gives \( R \sim M^{-1/3} \), so the larger the mass, the smaller the radius, and vice versa. In fact, this is what we see in white dwarf stars, which are well described by a degenerate equation of state, and which exhibit an inverse correlation between mass and radius. This mass-radius relation also applies to the cores of lower-mass main sequence stars, which are usually degenerate.

As we look at more and more massive white dwarfs, with decreasing radius they eventually become so dense inside that they are better described by a relativistic degenerate equation of state,

\[
P = 1.2435 \times 10^{15} \left( \frac{\rho}{\mu_e} \right)^{4/3}.
\] (25)

This corresponds to an \( n = 3 \) polytrope with \( K = 1.2435 \times 10^{15} \mu_e^{-4/3} \). Here, things get interesting: with \( n = 3 \), equation (24) formally indicates that the radius becomes undefined. If we go back to the preceding equation, and substitute in \( n = 3 \), we can see the radius drops out completely, allowing us to solve for the mass as

\[
M = 4\pi \left( \frac{K}{\pi G} \right)^{3/2} \left( -z_s^2 \frac{dw}{dz} \right)_{z=z_s}.
\] (26)

Substituting in the above value for \( K \), and using the fact that \( z_s^2 w'(z_s) = -2.02 \) for an \( n = 3 \) polytrope (see above), we find that in solar units the mass is given by

\[
M = \frac{5.836}{\mu_e} M_\odot.
\] (27)

This is the Chandrasekhar limiting mass. For white dwarf stars with masses less than this value, the equation of state will be somewhere between non-relativistic (22) and relativistic (25). As material is added (for instance, from a binary-star companion), the radius will shrink, and the density increase, until the completely-relativistic state is reached at the limiting mass. Beyond this mass, no hydrostatic equilibrium can be reached — in fact, the star becomes unstable, and will collapse and then explode to form a type-I supernova.