1. **Ionization fraction in \( \log T \) vs. \( \log \rho \) plane, overplotted with structure for Present, RG, and HB phases of sun**

   a. Assuming for simplicity a pure Hydrogen stellar envelope, apply the Saha equation given §14.2 of notes to derive for yourself the expression for the fraction \( f = n_+ / n_H \) of ionized hydrogen as a function of temperature \( T \) and total hydrogen number density \( n_H = n_+ + n_0 \), where \( n_+ \) and \( n_0 \) are respectively the number densities of ionized and neutral hydrogen.

   b. Assume the ratio of the ionized to neutral partition function is \( g_+/g_0 = 1/2 \). Evaluate all the constants in the above expression to convert it to a numerical form that depends symbolically only on the temperature \( T \) and mass density \( \rho \) (in CGS units).

   c. Plot contours for \( f = 0.1, 0.5, \) and \( 0.9 \) in the \( \log T \) vs. \( \log \rho \) plane, using the range \( 3.5 < \log T < 6 \) and \( -13 < \log \rho < 2 \). For future use, save this plot, e.g. with the name p0.

   d. Using the *Mathematica* notebook for solar structure you derived in the previous problem set, plot \( \log T \) vs. \( \log \rho \) for the sun at roughly its current age of \( \sim 4.5 \) Byr, selecting the blue color. Saving this plot as p1, next make and save two similar plots, colored red and green, for Red Giant and Horizontal Branch phases just before and after the He flash (e.g., time step numbers 241 and 259). Save these respectively as p2 and p3.

   e. Now use Show[p0,p1,p2,p3] to make an overplot Are the RG densities/temperature for H-ionization higher or lower than for the MS? In terms of stellar evolution and the Saha equation, briefly explain the differences. Make an overplot of this with results from part c, and use this estimate the temperature and density at which \( f = 1/2 \) for the present, RG, and HB stages of the sun.

   a, b:

   sec. 14.2, Saha eqn.: 

   \[
   \frac{n_{i+1}}{n_i} = \frac{g_{i+1}}{g_i} \frac{2}{n_e} \left( \frac{2\pi m e}{kT} \right)^{3/2} e^{-\Delta E_i/kT}.
   \]

   Let \( i = 0 \), so that \( f = n_+/n_H = n_e/\n_H \), and \( n_0 = n_H (1-f) \). Then Saha eqn. can be rewritten:

   \[
   \frac{\rho}{1-f} = \frac{g_e}{g_0} 2 m_p \left( \frac{2\pi m_e k}{n^2} \right)^{3/2} \frac{T^{3/2}}{\rho} e^{-\Delta E_e/kT} = 4.0 \times 10^{-6} \frac{T^{3/2}}{\rho} e^{-1.6 \times 10^4/T} \equiv b(\rho,T)
   \]

   From the quadratic formula, we then have:
\[ f = \frac{1}{2} (-b + \sqrt{b^2 + 4b}) \]

\textbf{c: coding this in Mathematica gives:}

\[
\begin{align*}
b[\rho, T] & := 4.0 \times 10^{-6} \frac{T^{3/2}}{\rho} \, e^{-1.6 \times 10^5 / T} \quad (* \text{MacDonald notes 07-IonRecomb eqn. 7.5.5} *) \\
f[\rho, T] & := 0.5 \left(-b[\rho, T] + \sqrt{b[\rho, T]^2 + 4b[\rho, T]} \right) \\
f[l\rho, lT] & := f[10^{1\rho}, 10^{1T}] \\
p0 & = \text{ContourPlot}[f[l\rho, lT], \{l\rho, -10, 4\}, \\
& \quad \{lT, 3.6, 6.5\}, \text{ContourShading} \rightarrow \text{False}, \text{Contours} \rightarrow \{.1, .5, .9\}, \\
& \quad \text{ContourStyle} \rightarrow \{(\text{Thick, Dotted}), \{\text{Medium, DotDashed}\}, \text{Thick}\}, \text{ContourLabels} \rightarrow \text{True}, \\
& \quad \text{FrameLabel} \rightarrow \{"\text{Log } \rho \ (\text{kg/m}^3)"\, , \"\text{Log } T \ (K)"\}\}, \text{GridLines} \rightarrow \text{Automatic}]
\end{align*}
\]
d, e: Overplot of $T$ vs. $\rho$ for present, RG, and HB phases of the sun is:

\[
\text{Show}\left[p0, p123, \text{PlotLabel} \rightarrow \{"f = H \text{ ion fraction}; \text{present sun, RG, HB}"\}\right]
\]

Reading dot-dashed intersections give temperature and density for $f=1/2$ as:

- **Present day:** $\log T=4.4$, $\log \rho=-1.5$ kg/m$^3$
- **HB:** $\log T=4.3$, $\log \rho=-3.9$ kg/m$^3$
- **RG:** $\log T=4.1$, $\log \rho=-4.3$ kg/m$^3$
2. Stellar evolution with EZ-Web for $M=10$ Msun

2. a.-d Repeat parts a-d of Problem 4 of HW7, but for a star of mass $M = 10M_\odot$.

e. Using “Manipulate toolbar”, plot mass $M$ vs. radius $r$ at time steps $t=1$ and $t=400$, corresponding to the ZAMS and the final stage of the star’s life.

f. Do the same for temperature $T$ vs. mass $M$, and then use the “additional plots” section to make overplots comparing $T$ vs. $M$ at both these ages.

g. Next, make an overplot of $\log \nabla_{ad}$ and $\log \nabla_{rad}$ vs. $r$ for each of these times. From this, estimate the radius range of the star’s convection zone (if any) at each stage.

h. Finally, make an analogous overplot for $\log \nabla_{ad}$ and $\log \nabla_{rad}$ vs. $M$ for these times. From this, estimate the mass range of the star’s convection zone, if any. For each stage, what does this imply about the nature of energy transport though the most of stars material envelope from core to surface?

soln

- e. $M$ vs. $r$ for $t=1$ and $t=400$
f. Overplots of $T$ vs. $M$ for $i=1$ (black) and $i=400$ (blue), on both linear and log scales for $T$:

\[ \text{[t, 2.56559 \times 10^7 \text{ yr}]} \]

\[ \text{[t, 2.56559 \times 10^7 \text{ yr}]} \]

\[ \text{[t, 2.56559 \times 10^7 \text{ yr}]} \]

\[ \text{[t, 2.56559 \times 10^7 \text{ yr}]} \]

g. Overplots of $\log \nabla_{\text{ad}}$ (blue) and $\log \nabla_{\text{rad}}$ (purple) vs. $r$ for $i=1$ and $i=400$:
Initial ZAMS model has $\nabla_{\text{rad}} > \nabla_{\text{ad}}$ only in core, $r < 1$ Rsun

Final stage model has $\nabla_{\text{rad}} > \nabla_{\text{ad}}$ in both small core $r < 1$ Rsun, and envelope $R > 1.5$ Rsun

h. Overplots of $\log \nabla_{\text{ad}}$ (blue) and $\log \nabla_{\text{rad}}$ (purple) vs. $M$ for $\text{it}=1$ and $\text{it}=400$: 
Initial ZAMS model has $\nabla_{\text{rad}} > \nabla_{\text{ad}}$ only in core, $M < 3.5$ Msun

Final stage model has $\nabla_{\text{rad}} > \nabla_{\text{ad}}$ in both innermost core $M < 0.1$ Msun, and envelope $M > 3.2$ Msun
3. **log T vs. log ρ domain diagram for ionization, degeneracy, etc.**

3. The goal of this problem is derive expressions for various transitions for the equation of state for a pure hydrogen gas, and plot these as lines in the log-temperature vs. log-density plane.

   a. To begin, derive an expression for log T at which, for a given log ρ, the gas and radiation pressure are equal, assuming the H is fully ionized. Plot this on a y=log-T vs. x=log-ρ diagram.

   b. In the semi-classical picture of orbital size, pressure ionization occurs when there is more than one atom in a sphere with radius equal to the Bohr radius. Derive the associated critical mass density for such pressure ionization ρ_{pi}, and plot this as vertical line in the log-T vs. log-ρ plane.

   c. Next derive expressions for log ρ at which, for a given log T, the hydrogen ionization fraction is f=0.1, 0.5, and 0.9. Again plot this on a y=log-T vs. x=log-ρ diagram, truncating the curves for densities above the pressure ionization value ρ_{pi} (In Mathematica, reversing x-y axes can be done using ParametricPlot.)

   d. Next derive the log T at which, for a given log ρ, the expressions for pressure for a (non-degenerate) ideal gas equals that for the pressure of a (non-relativistic) degenerate electron gas. Plot this vs. log ρ.

   e. Now derive the log ρ for the relativistic and non-relativistic expression for degenerate electron pressure are equal. As in part b, plot this as vertical line in the log-T vs. log-ρ plane.

   f. Finally, make an overplot of the plots from parts a.-e., choosing the wide ranges 3 < log T < 9 and -10 < log ρ < 8 in temperature and density (cgs units). (In Mathematica, this can be done with Show[pa, pb, pc, pd, pe], but you will need to ensure that each original plot covers these desired ranges.)

   g. Print out this master overplot, and label (either by hand or via computer editing) regions where:

   1. $P_{rad} > P_{gas}$ and vice versa.
   2. Hydrogen is mostly ionized, partly ionized, and mostly neutral.
   3. The gas is non-degenerate vs. degenerate but non-relativistic.
   4. The degenerate gas is non-relativistic vs. relativistic.

   No need to submit hardcopies of the individual source figures pa, pb, etc. Just give the formulae for parts a-e, and the final master plot.

---

**Calculations**

**Final results: formulae for a-e, and annotated “master plot”**
Final results:

a. \( P_{\text{rad}} = P_{\text{gas}} \):
\[
\log T = 7.6 + \frac{1}{3} \log \rho
\]

b. Pressure ionization occurs at:
\[
\log \rho_{\text{pi}} = 0.433
\]

c. Ionization fractions set by:
\[
\log \rho_i = C + \frac{3}{2} \log\left(\frac{T}{10^5 \text{K}}\right) - \left(\frac{10^5 \text{K}}{T}\right)
\]
where \( C = \{0.36, -1.29, -2.50\} \) for \( f = \{0.1, 0.5, 0.9\} \)

d. Equal pressure for ideal gas and non-relativistic degenerate gas requires:
\[
P_{\text{gas}} = 2 n_e k T = P_{\text{nr}} = n_e^{\frac{5}{3}} \frac{h^2}{m_e}
\]
Solving gives critical temperature for degeneracy as a function of density:
\[
T_{\text{dgnr}} = \left(\frac{\rho}{m_p}\right)^{\frac{2}{3}} \frac{h^2}{(2 m_e k)}
\]

e. Equal pressure for relativistic and non-relativistic degenerate gas requires:
\[
P_{\text{rel}} = n_e^{\frac{4}{3}} \frac{h c}{m_e} = P_{\text{nr}} = n_e^{\frac{5}{3}} \frac{h^2}{m_e}
\]
Solving gives critical density for relativistic degeneracy:
\[
\rho_{\text{rel}} = \left(\frac{m_e c}{h}\right)^{\frac{3}{2}} m_p ; \quad \log[\rho_{\text{rel}}] = 7.46
\]

Putting this all together, here then is an annotated “master plot” of boundaries in the log \( T \) vs. log \( \rho \) plane:
4. EZWeb summary results for M=0.1 to 100 Msun

c. With the data in hand, plot the luminosity log $L$ (from column 4 of the summary files) vs. the log of $4\pi R^2 \sigma T_{eff}^4$, where $R$, and $T_{eff}$ are respectively radius and effective temperature (from columns 5 and 6), and $\sigma$ is the Stefan-Boltzmann constant. Note that EZ-Web gives luminosity and radius in solar units, $L_{\odot}$ and $R_{\odot}$. Confirm that the plotted curve is indeed a straight line with unit slope.
d. From the resulting table of summary.txt file data, create plots of
   - log $L/L_\odot$ vs. log $M/M_\odot$
   - log $R/R_\odot$ vs. log $M/M_\odot$
   - log $T_{eff}$ vs. log $M/M_\odot$

Make sure to clearly label the axes.
Red dotted line, which has slope 3 in log-log plot that is predicted by scaling analysis in sec. 16.2, fits the points quite well.
For Mass-Radius plot, the 3 lines are for slopes of 0.5, 0.75 and 1.
The steepest, unit-slope line nearly fits the data for lower-mass stars with M<1 Msun.
The shallowest, 1/2-slope line near fits the data for higher-mass stars with M > 1 Msun.