PHYS-633: Introduction to Stellar Astrophysics

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1. Course Overview: Stellar Atmospheres vs. Interiors

A solid understanding of stars and stellar evolution must include both a study of the structural principles of their interiors as well as the properties and conditions in their surface atmosphere that sets their observable characteristics. The goals of this course are to develop a solid understanding of both the atmospheres and interiors of stars.

While the atmosphere consists of only a tiny fraction of the overall stellar radius and mass (respectively about $10^{-3}$ and $10^{-12}$), it represents a crucial boundary layer between the dense interior and the near vacuum outside, from which the light we see is released, imprinting it with detailed spectral signatures that, if properly interpreted in terms of the physics principles coupling gas and radiation, provides essential information on stellar properties. In particular, the identities, strengths, and shapes (or profile) of spectral lines contain important clues to the physical state of the atmosphere – e.g., chemical composition, ionization state, effective temperature, surface gravity, rotation rate. However these must be properly interpreted in terms of a detailed model atmosphere that accounts properly for basic physical processes, namely: the excitation and ionization of atoms; the associated absorption, scattering and emission of radiation and its dependence on photon energy or frequency; and finally how this leads to such a complex variation in emitted flux vs. frequency that characterizes the observed spectrum. Such model atmosphere interpretation of stellar spectra forms the basis for inferring basic stellar properties like mass, radius and luminosity.

But once given these stellar properties, a central goal is to understand how they interrelate with each other, and how they develop and evolve in time. The first represents the problem of stellar structure, that is, the basic equations describing the hydrostatic support against gravity, and the source and transport of energy from the deep interior to the stellar surface. The latter problem of stellar evolution breaks naturally into questions regarding the origin and formation of stars, their gradual aging as their nuclear fuel is expended, and how they eventually die.

The first issue regarding origin and formation of stars represents a very broad area of active research; it could readily encompass a course of its own, but is usually part of a course on the Inter-Stellar Medium (ISM), which indeed provides the source mass, from gas and dust in dense, cold molecular clouds, that is condensed by self gravity to form a proto-star. Apart from some brief discussions, e.g. of the critical “Jean’s” length or mass for gravitational collapse of a cloud, we will have little time for discussion of star formation in this course.

Instead, our examination of stellar evolution will begin with the final phases of contrac-
tion (on what’s known as the “Hayashi track”) toward the Zero-Age Main Sequence (ZAMS), representing when the core of a star is first hot enough to allow a nuclear-fusion burning of hydrogen into helium. This then provides an energy source to balance the loss by the radiative luminosity, allowing the star to remain on this MS for many millions or even (e.g. for the Sun) billions of years. But the gradual MS aging as this core-H fuel is spent leads ultimately to the Terminal Age Main Sequence (TAMS), when the core H is exhausted, whereupon the star actually expands to become a cool, luminous “giant”.

A key overall theme is that the luminosity and lifetime on the MS, and indeed the very nature of the post-MS evolution and ultimate death of stars, all depend crucially on their initial mass.

1.1. Differences between Atmospheres and Interiors

But our path to this heart of a star and its life passes necessarily through the atmosphere, the surface layers that emit the complex radiative spectrum that we observe and aim to interpret. So let us begin by listing some key differences between stellar atmospheres vs. interiors:

- **Scale:** Whereas a stellar interior extends over the full stellar radius $R$, the atmosphere is just a narrow surface layer, typically only about $10^{-3} R$.

- **Mean-Free-Path:** By definition an atmosphere is where the very opaque nature of the interior finally becomes semi-transparent, leading to escape of radiation. This transparency can be quantified in terms of the ratio of photon mean-free-path $\ell$ to a characteristic length to escape. In the interior $\ell$ is very small (a few cm), much smaller than the stellar radius scale $R$. But in the atmospheric surface $\ell$ becomes of comparable to the scale height $H$ (which is also of order $10^{-3} R$), over which the density and pressure drop by a factor $e$. Upward-propagating photons with local $\ell > H$ thus find themselves in an even less dense medium with even longer $\ell$, and so effectively can escape entirely from the star.

- **Isotropy:** The opaqueness of the interior means its radiation field is nearly isotropic, with only a tiny fraction (of order $\ell/R \ll 1$) more photons going up than down. By contrast, the escape from the photosphere makes the radiation field there distinctly anisotropic, with most radiation going up, and little or none coming down from the nearly empty and transparent space above.
• **Radiation Transport**: Given the above distinctions, we can see that the transport of radiative energy out of the interior can be well modeled in terms of a *local diffusion*; by contrast, the transition to free escape in the atmosphere must be cast in terms of a more general equation of radiative transfer that in principle requires an inherently *non-local* solution involving spatial integrals over the emitting volume.

• **LTE vs. NLTE**: In the interior extensive absorption of radiation implies a strong radiation-matter coupling that is much like a classical blackbody, and so leads very nearly to a *Local Thermodynamic Equilibrium* (LTE), with the radiation field well characterized by the Planck function $B_\nu(T)$. In contrast, the inherently non-local transport in an atmosphere, particularly in a case with a strong degree of *scattering* that does not well couple the radiation to the thermal properties of the gas, can lead to a distinctly Non Local-Thermodynamic-Equilibrium 1 (NLTE).

• **Temperature**: The escape of radiation makes an atmospheric surface relatively “cool”, typically $\sim 10^4$ K, in contrast to the $\sim 10^7$ K temperature of the deep interior, where the opaqueness of overlying matter acts much like a very heavy “blanket”.

• **Pressure**: To balance its own self-gravity, a star has to have much higher pressure in the interior than at the surface.

• **Density**: Even with the high temperature, this gravitational compression also leads to a much higher central density.

• **Energy Balance**: While the surface luminosity represents energy *loss* from the atmosphere, the very high density and temperature within the deep stellar core results in a nuclear burning that provides an energy *source* to the interior.

Despite these many important differences, there are a few fundamental concepts that are essential for both atmospheres and interiors. The next few sections cover some key examples, for example the *opacity* that couples radiation to matter, and the balance between pressure gradient and gravity the supports both a star and its atmosphere in *hydrostatic equilibrium*.

1 The negation of the “Non” here is of the whole concept of LTE, and not, e.g., a TE that happens to be non-local (NL).
2. Escape of Radiation from a Stellar Envelope

Most everything we know about a star comes from studying the light it emits from the visible surface layer called the atmosphere. But the energy for this visible emission can be traced to nuclear fusion in the stellar core. As illustrated in fig. 2.1, to reach the surface, the associated photons must diffuse outward via a “random walk” through the stellar envelope.

To characterize the extent of this diffusion, let us estimate a typical photon mean-free-path (mfp) in the stellar interior,

\[ \ell \equiv \frac{1}{n\sigma} = \frac{1}{\kappa \rho} \tag{2.1} \]

Here \( n \) is the number density of absorbing particles (CGS units \( \text{cm}^{-3} \)), and \( \sigma \) is the effective cross section (CGS units \( \text{cm}^2 \)) it presents for absorbing radiation. Often in astrophysics it is more convenient to write the density in terms of the mass-per-unit volume \( \rho \) (with CGS units \( \text{g/cm}^3 \)), which can be related to the number density through a mean mass per particle \( \mu = \rho/n \). In this case it is also convenient to define the opacity \( \kappa = \sigma/\mu \) (with CGS units \( \text{cm}^2/\text{g} \)), which effectively represents the cross section per unit mass of absorbing material. This allows one to write the mean-free-path in terms of the mass density, as given in the second equality of (2.1).

2.1. Sources of Stellar Opacity

The opacity of stellar material is a central issue for both interiors and atmospheres, so let’s briefly summarize the sources of stellar opacity in terms of the basic physical processes.

For solid objects in our everyday world, the interaction with light depends on the object’s physical projected area. For example, for a solid sphere of radius \( a \) that is much larger than the wavelength \( \lambda \) of impinging light, the cross section is just\(^2\) \( \sigma = \pi a^2 \).

For atoms, ions and electrons that make up a gaseous object like a star, the effective cross sections for interaction with light depend on the microscopic atomic processes at hand. Because light is an Electro-Magnetic (EM) wave, its fundamental interaction with matter occurs through the variable acceleration of charged particles by the varying electric field in the wave. As the lightest common charged particle, electrons are most easily accelerated, and thus they are generally key for the interaction of light with matter. But the details of \footnote{For dust grains with \( a \lesssim \lambda \), the cross section declines with wavelength, often characterized as a power-law \( \sigma \sim \lambda^{-m} \), where the power index can be over the range \( m = 1 \) to \( 4 \) depending on the details. Absorption by interstellar dust thus results in a distinct reddening of a star’s color.}

\footnote{For dust grains with \( a \lesssim \lambda \), the cross section declines with wavelength, often characterized as a power-law \( \sigma \sim \lambda^{-m} \), where the power index can be over the range \( m = 1 \) to \( 4 \) depending on the details. Absorption by interstellar dust thus results in a distinct reddening of a star’s color.}
the interaction, and thus the associated cross sections and opacities, depend on the bound vs. free nature of the electron. The relevant combinations are illustrated in figure 2.2, and are briefly listed as follows:

1. Free electron (Thomson) scattering
2. Free-free (f-f) absorption
3. Bound-free (b-f) absorption
4. Bound-bound (b-b) absorption or scattering

Because a completely isolated electron has no way to store both the energy and momentum of a photon, it cannot by itself absorb radiation, and so instead simply scatters, or redirects it. Such “Thomson scattering” by free electrons (item #1) is thus distinct from the “true” absorption of the free-free (item #2) interaction of electrons that are unbound, but near enough to an ion to share the momentum and energy of the absorbed photon. In the latter, the electron effectively goes from one hyperbolic orbital energy to another. Since
Fig. 2.2.— Illustration of the free-electron and bound-electron processes that lead to scattering, absorption, and emission of photons.

the hyperbolic orbital energies of such free electrons are not quantized, such f-f absorption can thus in principal occur for any photon energy.

Moreover, for photons with sufficient energy to ionize or kick off an electron bound to an atom or ion, there is then also bound-free absorption (item #3). This is non-zero for any energy above the ionization threshold energy.

Lastly, for photons with just the right energy to excite an atom or ion from one discrete bound level to a higher bound level, there can be bound-bound absorption (item #4). This is the basic process behind the narrow absorption lines in observed stellar spectra. Following such initial absorption to excite an electron to a higher bound level, quite often the atom will then simply spontaneously de-excite back to the same initial level, emitting a photon of nearly the same original energy, but in a different direction, representing then a b-b scattering. On the other hand, if before this spontaneous decay can occur, another electron collisionally de-excites the upper level, then overall there is a “true” absorption, with the
energy of the photon ending up in the colliding electron, and thus ultimately shared with the gas.

A key distinction between scattering and true absorption is thus that the latter provides a way to couple the energetics of radiation and matter, and so tends to push both towards thermodynamical equilibrium (TE).

Note finally that the emission of radiation generally occurs by the inverse of the last 3 absorption processes in the above list, known thus as

1. Free-free (f-f) emission
2. Free-Bound (f-b) emission
3. Bound-bound (b-b) emission

### 2.2. Thomson Cross-Section and the Opacity for Electron Scattering

In the deep stellar interior, where the atoms are generally highly ionized, a substantial component of the opacity comes just from the scattering off free electrons. This is a relatively simple process that can be well analyzed in terms of classical electromagnetism, in contrast to the quantum mechanical models needed for photon interaction with electrons that are bound within atoms.

The upper left panel of figure 2.2 illustrates how the acceleration of an electron induces a corresponding variation of its own electric field, which then induces a new electromagnetic wave, or photon, that propagates off in a new direction. The overall process is called “Thomson scattering”, after J.J. Thomson, who in the late 19th century first worked out the associated ‘Thomson cross section’ using Maxwell’s equations for E&M. Details can be found in chapter 3 of Rybicki & Lightman (1978), but the site here gives a basic summary of the derivation: [http://farside.ph.utexas.edu/teaching/em/lectures/node96.html](http://farside.ph.utexas.edu/teaching/em/lectures/node96.html)

Intuitively, the result can be roughly understood in terms of the so-called “classical electron radius” $r_e$. This is obtained through the equality,

$$\frac{e^2}{r_e} \equiv m_e c^2,$$

where again $e$ is the magnitude of the electron charge, $m_e$ is the electron mass, and $c$ is the speed of light. The left side is just the classical energy needed to assemble the total electron charge within a uniform sphere of radius $r_e$, while the right side is the rest mass energy of an
electron from Einstein’s famous equivalence formula between mass and energy. Eqn. (2.2) can be trivially solved for the associated electron radius,

\[ r_e = \frac{e^2}{m_e c^2} \]  

(2.3)

In these terms, the Thomson cross section for free-electron scattering is just

\[ \sigma_{Th} = \frac{8}{3} \pi r_e^2 = 0.66 \times 10^{-24} \text{cm}^2 = 2/3 \text{ barn} \]  

(2.4)

(The term “barn” represents a kind of physics joke, because compared to the cross sections associated with atomic nuclei, it is a huge, “as big as a barn door”.) Thus we see that Thomson scattering has a cross section just 8/3 times greater than the projected area of a sphere with the classical electron radius.

For stellar material to have an overall neutrality in electric charge, even free electrons must still be loosely associated with corresponding positively charged ions, which have much greater mass. Defining then a mean mass per free electron \( \mu_e \), we can write the electron scattering opacity \( \kappa_e \equiv \sigma_{Th}/\mu_e \). Ionized hydrogen gives one proton mass \( m_p \) per electron, but for fully ionized helium (and indeed for most all heavier ions), there are two proton masses for each electron. For fully ionized stellar material with hydrogen mass fraction \( X \), we find then that

\[ \mu_e = \frac{2m_p}{1 + X} \]  

(2.5)

Since \( m_p \approx 5/3 \times 10^{-24} \text{ g} \), we thus obtain

\[ \kappa_e = 0.2 (1 + X) \frac{cm^2}{g} = 0.34 \text{cm}^2/g \]  

(2.6)

where the last equality applies a “standard” solar hydrogen mass fraction \( X = 0.72 \).

Two particularly simple properties of electron scattering are: 1) it is generally almost spatially constant, and 2) it is “gray”, i.e. independent of photon wavelength. This contrasts markedly with many other sources of opacity, which can depend on density and temperature, as well as on wavelength, particularly for line-absorption between bound levels of an atom. We discuss such other opacity source and their scalings in greater detail below. See also the associated sections on opacity by Rich Townsend and Jim MacDonald.
2.3. Random-Walk Model for Photon Diffusion from Stellar Core to Surface

For a star with mass \( M \) and radius \( R \), the mean stellar density is \( \bar{\rho} = \frac{M}{(4/3)\pi R^3} \), which for the sun with \( M_\odot \approx 2 \times 10^{33} \) g and \( R_\odot \approx 7 \times 10^{10} \) cm works out to be

\[
\bar{\rho}_\odot = \frac{M_\odot}{4\pi R_\odot^3/3} \approx 1.4 \text{ g/cm}^3,
\]

i.e. just above the density of water. Multiplying this by the electron opacity and taking the inverse then gives an average mean-free-path from electron scattering in the sun,

\[
\bar{\ell}_\odot = \frac{1}{0.34 \times 1.4} = 2 \text{ cm}.
\]

(2.7)

Of course, in the core of the actual sun, where the mean density is typically a hundred times higher than this mean value, the mean-free-path is yet a factor hundred smaller, i.e \( \ell_{\text{core}} \approx 0.2 \) mm!

But either way, the mean-free-path is much, much smaller than the solar radius \( R_\odot \approx 7 \times 10^{10} \) cm, implying a typical optical depth,

\[
\tau = \frac{R_\odot}{\bar{\ell}_\odot} \approx 3.5 \times 10^{10}.
\]

(2.9)

The total number of scatterings needed to diffuse from the center to the surface can then be estimated from a classical “random walk” argument. The simple 1D version states that after \( N \) left/right random steps of unit length, the root-mean-square (rms) distance from the origin is \( \sqrt{N} \). For the 3D case of stellar diffusion, this rms number of unit steps can be roughly associated with the total number of mean-free-paths between the core and surface, i.e. \( \tau \). This implies that photons created in the core of the sun need to scatter a total of \( \tau^2 \approx 10^{21} \) times to reach the surface! Along the way the cumulative path length travelled is \( \ell_{\text{tot}} \approx \tau^2 \bar{\ell}_\odot \approx \tau R_\odot \). For photons travelling at the speed of light \( c = 3 \times 10^{10} \text{ cm/s} \), the total time for photons to diffuse from the center to the surface is thus

\[
t_{\text{diff}} = \tau^2 \bar{\ell}_\odot c \approx \tau R_\odot c = 3.5 \times 10^{10} \times 2.3 \text{ s} \approx 2600 \text{ yr},
\]

(2.10)

where for the last evaluation, it is handy to note that 1 yr \( \approx \pi \times 10^7 \) s.

Once the photons reach the surface, they can escape the star and travel unimpeded through space, taking, for example, only a modest time \( t_{\text{earth}} = au/c \approx 8 \text{ min} \) to cross the 1 au distance from the sun to the earth. The stellar atmospheric surface thus marks a quite distinct boundary between the interior and free space. From deep within the interior, the stellar radiation field would appear nearly isotropic (same in all directions), with only a small
asymmetry (of order $1/\tau$) between upward and downward photons. But near the surface, this radiation becomes distinctly anisotropic, emerging upward from the surface below, but with no radiation coming downward from empty space above.

We shall see below that this atmospheric transition between interior and empty space occurs over a quite narrow layer, typically a few hundred kilometers or so, or about a thousandth of the stellar radius.

3. Density Stratification from Hydrostatic Equilibrium

In reality of course stars are not uniform density, because the star’s self-gravity attracts material into a higher inward concentration. The total mass inside a given radius $r$ can be derived from the volume integral over the spatially varying density $\rho$,

$$M_r \equiv \int_{0}^{r} 4\pi \rho(r') r'^2 \, dr'.$$

(3.1)

This leads then to a local inward gravitational acceleration

$$g = \frac{GM_r}{r'^2}.$$

(3.2)

The associated inward gravitational force-per-unit volume $\rho g$ must be balanced by a (negative) radial gradient in the gas pressure $P$,

$$\frac{dP}{dr} = -\rho \, g = -\rho(r) \frac{GM_r}{r^2},$$

(3.3)

a condition known as Hydrostatic Equilibrium. This is one of the fundamental equations of stellar structure, with important implications for the properties of both the interior and atmosphere. For the atmosphere, the mass, radius, and gravity are fixed at surface values $M$, $R$, and $g_* = GM/R^2$; but in the interior their radial variation must be taken into account.

To relate the density and pressure, a key auxiliary equation here is the Ideal Gas Law, which in this context can be written in the form,

$$P = \rho \frac{kT}{\bar{\mu}},$$

(3.4)

where $k = 1.38 \times 10^{-16}$erg/K is Boltzmann’s constant, $T$ is the temperature, and $\bar{\mu}$ is the average mass of all particles (e.g. both ions and electrons) in the gas. For any given element,
the fully ionized molecular weight is just set by the ratio of the nuclear mass to charge. For fully ionized mixture with mass fraction $X$, $Y$, and $Z$ for H, He, and metals, the overall mean molecular weight comes from the inverse of the inverse sum of the individual molecular weights,

$$\bar{\mu} = \frac{m_p}{2X + 3Y/4 + Z/2} \approx 0.6m_p$$  \hspace{1cm} (3.5)

where the last equality is for the solar case with $X \approx 0.72$, $Y \approx 0.26$, and $Z \approx 0.02$.

The ratio of eqns. (3.4) to (3.3) defines a characteristic scale height for variations in the gas pressure,

$$H \equiv \frac{P}{|dP/dr|} = \frac{kT}{\bar{\mu}g} = \frac{kTr^2}{\bar{\mu}GM_r}.$$  \hspace{1cm} (3.6)

The huge differences in temperature between the interior vs. surface imply a correspondingly large differences in the pressure scale height $H$ in the stellar envelope vs. atmosphere.

### 3.1. Thinness of Atmospheric Surface Layer

While the temperatures in the deep interior of a star can be many million Kelvin, the characteristic temperature in the atmospheric surface layers of a star are typically of order a few thousand Kelvin, i.e. near the stellar effective temperature, which for a star of luminosity $L$ and radius $R$ is given by

$$T_{\text{eff}} = \left[ \frac{L}{4\pi\sigma R^2} \right]^{1/4} = 5800 K \left( \frac{L/L_\odot}{R/R_\odot} \right)^{1/4},$$  \hspace{1cm} (3.7)

where (as discussed in sec. 5.2 of DocOnotes1-stars) $\sigma$ here is the Stefan-Boltzmann constant. For the solar surface gravity $g_{\text{grav}} = GM_\odot/R_\odot^2 \approx 2.7 \times 10^4 \text{ cm/s}^2$, molecular weight $\bar{\mu} \approx 0.6m_p \approx 10^{-24} \text{ g}$, and photospheric temperature $T_\odot = 5800 \text{ K}$, we thus obtain a surface pressure scale height that is only a tiny fraction of the solar radius,

$$H_\odot \approx 0.0005 R_\odot \approx 300 \text{ km}.$$  \hspace{1cm} (3.8)

This relative smallness of the atmospheric scale height is a key general characteristic of static stellar atmospheres, common to all but the most extremely extended giant stars. In general, for stars with mass $M$, radius $R$, and surface temperature $T$, the ratio of scale height to radius can be written in terms of the ratio of the associated sound speed $a_* \equiv \sqrt{kT/\bar{\mu}}$ to surface escape speed $v_e \equiv \sqrt{2GM/R}$,

$$\frac{H}{R} = \frac{2a_*^2}{v_e^2} \approx 5 \times 10^{-4} \frac{T}{T_\odot} \frac{R/R_\odot}{M/M_\odot}.$$  \hspace{1cm} (3.9)
For the solar surface, the sound speed is $a_s \approx 9$ km/s, about $1/70$th of the surface escape speed $v_e = 620$ km/s.

Because of this relative thinness of atmospheric scale heights, the emergent spectrum from a star can generally be well modeled in terms of a 1D planar atmosphere, in which local conditions vary only with vertical height $z = r - R$. While this height is measured relative to some reference layer $z = 0$ near the stellar radius $R$, the characteristics of the model are themselves largely independent of $R$.

Within the atmosphere itself, the scale length for variation in temperature, $T/|dT/dr|$, is typically much larger than a pressure scale height $H$. If we thus approximate the atmosphere as being roughly isothermal with a constant temperature $T \approx T_{\text{eff}}$, then both the density and pressure will have an exponential stratification with height $z = r - R$,

$$\frac{P(z)}{P_*} \approx \frac{\rho(z)}{\rho_*} \approx e^{-z/H},$$

where the asterisk subscripts denote values at the surface layer where $z \equiv 0$.

Once the density drops to a level where the photon mean-free-path becomes comparable to this relatively small atmospheric scale height, radiation no longer diffuses, but rather escapes fully into an unimpeded propagation away from the star. This transition between diffusion and free escape occurs over just a few scale heights $H \ll R$. This explains the very sharp edge to the visible solar photosphere.

### 3.2. Hydrostatic balance in stellar interior and the Virial Temperature

This hydrostatic balance must also apply in the stellar interior, but a key difference is that, at any given interior radius $r$, the local gravitational acceleration depends only on the mass within that radius,

$$M(r) \equiv 4\pi \int_0^r \rho(r')r'^2 \, dr'.$$

This thus requires the hydrostatic equilibrium equation to be written in the somewhat more general form,

$$\frac{dP}{dr} = -\rho(r) \frac{GM(r)}{r^2}. $$

---

$^3$Analogously, the properties of earth’s atmosphere are quite sensitive to the height above sea level, which is at a central radial distance of roughly an earth’s radius. But otherwise, the actual value of earth’s radius plays little role in the physics of the atmosphere, which is also much more affected by earth’s gravity and characteristic surface temperature.
This represents one of the key equations for stellar structure.

The implications of hydrostatic equilibrium for the hot interior of stars are quite different from the steep exponential pressure drop near the surface; indeed they allow us now to derive a remarkably simple scaling relation for a characteristic interior temperature $T_{\text{int}}$.

For this consider the associated interior pressure $P_{\text{int}}$ at the center of the star ($r = 0$); to drop from this high central pressure to the near-zero pressure at the surface, the pressure gradient averaged over the whole star must be $|dP/dr| \approx P_{\text{int}}/R$. We can similarly characterize the gravitational attraction in terms of the surface gravity $g_* = GM/R^2$ times an interior density that scales as $\rho_{\text{int}} \sim P_{\text{int}} \bar{\mu}/kT_{\text{int}}$. Applying these in the basic definition of scale height (3.6), we find that for the interior $H \approx R$, which in turn implies for this characteristic stellar interior temperature,

$$T_{\text{int}} \approx \frac{GM\bar{\mu}}{kR} \approx 13 \times 10^6 K \frac{M/M_\odot}{R/R_\odot}. \quad (3.13)$$

Thus, while surface temperatures of stars are typically a few thousand Kelvin, we see that their interior temperatures are typically of order 10 million Kelvin! As discussed below (§19), this is indeed near the temperature needed for nuclear fusion of Hydrogen into Helium in the stellar core.

This close connection between thermal energy of the interior ($\sim kT$) to the star’s gravitational binding energy ($\sim GM\bar{\mu}/R$) is really just another example of the Virial theorem for gravitationally bound systems. The temperature is effectively a measure of the average kinetic energy associated with the random thermal motion of the particles in the gas. Thermal energy is thus just a specific form of kinetic energy, and the Virial theorem tells us that the average kinetic energy in a bound system equals one-half the magnitude of the gravitational binding energy. For details, see, e.g., R. Townsend’s Stellar Interior notes 01-virial.pdf, and the discussion in §16.3 below.

4. The Stellar Radiation Field

4.1. Surface Brightness or Specific Intensity $I$

The radiation field within a star can be fundamentally described in terms of the intensity, $I$. For radiation of frequency $\nu$ at some spatial location $\mathbf{r}$, $I_\nu(\hat{n}, \mathbf{r})$ represents the energy per-
unit-area, per-unit-time, per-unit-frequency, and per-unit solid angle\(^4\) about the radiation direction \(\hat{n}\). It can also be thought of the “surface brightness” of a small patch of the sky in a given direction.

For intensity in the direction of differential element of solid angle \(d\Omega\), the energy \(dE\) crossing through an area \(dA\) in a time \(dt\) and in a frequency range \(d\nu\) is

\[
dE = I_\nu \, dA \, dt \, d\Omega \, d\nu.
\]

(4.1)

As illustrated in figure 4.1, a key point is that the intensity remains unchanged during propagation through a vacuum, i.e. without any material to absorb or emit radiation. This may seem at first surprising because intuitively we know that the flux from a radiative source typically falls off with the inverse square of the distance. But note then the specific intensity represents a kind of flux per unit solid angle.

Since the solid angle of a source with fixed size, like the sun, also declines with inverse distance squared, the specific intensity of the resolved source, for example the surface brightness of the sun, remains the same viewed from any distance. When we see the sun in earth’s sky, its disk has the same brightness (ignoring absorption by earth’s atmosphere) as it would have if we were to stand on the surface of the sun itself!! See §5.1 of DocOnotes1-stars for some further discussion on the constancy of specific intensity and the meaning of solid angle.

In contrast to this constancy of specific intensity for radiation propagating through free space, the specific intensity at any point within the atmosphere or interior of a star depends on the sources and sinks of radiation from its interaction with stellar matter. As we shall see in the discussion below, this can in general quite difficult to determine. But within the very deep stellar interior, it again becomes relatively simple to describe, set by \(I_\nu \approx B_\nu\), where the Planck blackbody function depends only on the local temperature \(T\). Note this implies that \(I_\nu\) in the interior is isotropic, i.e. the same in all directions \(\hat{n}\).

Between these simple limits of the isotropic, Planckian intensity of the deep interior and the free-streaming constancy of intensity through empty space, lies the stellar atmosphere, where \(I_\nu\) is generally a complex function of spatial location \(r\), frequency \(\nu\), and direction.

\(^4\)A solid angle is a 2-D generalization of a 1-D angle in a plane. Much as the circumference of a unit circle implies there are \(2\pi\) radians along the full arc of the circle, so the area of a unit sphere implies that the solid angle of the full sky is \(4\pi\) steradians. If one imagines a sphere centered on the observer but outside some observed object, then the solid angle of the object is given by the area of the shadow the object makes on the sphere divided by the square of the radius of the sphere. For an object with an area \(A\) and surface normal that makes an angle \(\theta\) with the line of sight to observer, the solid angle seen from a large distance \(D\) is approximately \(\Omega = A \cos \theta / D^2\) steradian.
Fig. 4.1.— Left: The intensity $I_{em}$ emitted into a solid angle $\Omega_{rec}$ located along a direction that makes an angle $\theta$ with the normal of the emission area $A_{em}$. Right: The intensity $I_{rec}$ received into an area $A_{rec} = D^2 \Omega_{rec}$ at a distance $D$ from the source with projected solid angle $\Omega_{em} = A_{em} \cos \theta / D^2$. Since the emitted and received energies are equal, we see that $I_{em} = I_{rec}$, showing that intensity is invariant with distance $D$.

But if one ignores the complex 3-D spatial structure seen in actual views of, e.g. the solar atmosphere, the overall radial thinness of an atmosphere implies that the directional dependence can be completely described in terms of the projection of the radiation direction onto the local vertical, i.e. $\mu = \mathbf{n} \cdot \mathbf{z}$, giving then $I_{\nu}(\mu, z)$.

Moreover, if we defer for now discussion of the dependence on frequency, we can suppress the $\nu$ index, and so simply write $I(\mu, z)$. 

4.2. Mean Intensity $J$

In some contexts, it is of interest just to describe the angle-averaged intensity, that is, $I$ integrated over the full $4\pi$ steradians of solid angle $\Omega$,

$$J \equiv \frac{1}{4\pi} \int I(\hat{n}) d\Omega = \frac{1}{2} \int_{-1}^{+1} I(\mu) d\mu, \quad (4.2)$$

where the latter applies to the case of a planar atmosphere, noting that $d\Omega = -d\mu d\phi$, and carrying out the integral over the $2\pi$ radians of the azimuthal angle $\phi$, over which $I$ is assumed to be constant. In such a planar case, the mean intensity depends only on height, $J(z)$.

For light propagating at speed $c$, the mean intensity gives directly the radiative energy density per unit volume,

$$U = \frac{4\pi J}{c}. \quad (4.3)$$

4.3. Eddington Flux $H$

The vector radiative flux is given by

$$\mathbf{F} \equiv \int \hat{n} I(\hat{n}) d\Omega = 2\pi \hat{z} \int_{-1}^{+1} \mu I(\mu) d\mu \equiv 4\pi H \hat{z}, \quad (4.4)$$

where again the second equality applies for the vertical flux in a planar atmosphere. The last equality defines the Eddington flux,

$$H \equiv \frac{1}{2} \int_{-1}^{+1} \mu I(\mu) d\mu, \quad (4.5)$$

which is constructed to have an analogous form to the mean intensity $J$, but now with a $\mu$ factor within the integral.

4.4. Second Angle Moment $K$

The mean intensity and Eddington flux can also be characterized as the two lowest angle moments of the radiation field. For a planar atmosphere, the general “$j$-th” moment is defined by

$$M_j \equiv \frac{1}{2} \int_{-1}^{+1} \mu^j I(\mu) d\mu, \quad (4.6)$$
by which we see that $J$ and $H$ represent respectively the zeroth and first moments. The next highest, or second moment, is given by setting $j = 2$,

$$K \equiv \frac{1}{2} \int_{-1}^{+1} \mu^2 I(\mu) d\mu. \quad (4.7)$$

Physically $K$ is related to the *radiation pressure*, $P_R$. In a planar atmosphere, this just represents the *vertical* transport of the *vertical momentum* of radiation. Since radiative momentum is given by the energy divided by light speed, we find

$$P_R \equiv 4\pi \int_{-1}^{+1} \frac{I(\mu)}{c} \mu^2 d\mu = \frac{4\pi K}{c}. \quad (4.8)$$

![Fig. 4.2.— A diagram to illustrate how the surface intensity $I(\mu)$ relates to that seen by an observer at distance $r$ from a star of radius $R$, with the observer angle $\alpha$ and surface angle $\theta = \arccos \mu$ related through the law of signs.](image)

**Exercise:** As illustrated in figure 4.2, consider an observer at a distance $r$ from the center of a star of radius $R$ that has an uniformly bright surface, i.e., $I(\mu) = I_o = \text{constant}$ for all $\mu > 0$.

a. Derive analytic expressions for $J(r)$, $H(r)$, and $K(r)$ in terms of $\alpha_* \equiv \sin^{-1}(R/r)$.

b. Use these to write expressions for the ratios $H/J$ and $K/J$ in terms of $\alpha_*$, and in terms of $r/R$.

c. Plot both ratios vs. $r/R$ over the range [1, 5].
Exercise: The angular diameter of the sun is $\Delta \alpha = 30'$. Suppose that terrestrial atmospheric seeing effects limit our angular resolution to $\delta \alpha = 1''$.

a. Show that this sets a lower bound on the $\mu$ for which we can infer $I(\mu)$.

b. Derive a formula for this $\mu_{\text{min}}$ in terms of $\Delta \alpha$ and $\delta \alpha$.

c. Compute a numerical value for the solar/terrestrial parameters given above.

4.5. The Eddington Approximation for Moment Closure

We can in principle continue defining higher and higher moments, but their physical meaning becomes more and more obscure. Instead, it often useful to truncate this moment procedure by finding a closure relation between a higher and a lower moment, usually $K$ and $J$. It then becomes useful to define the ratio $f \equiv K/J$, which was first emphasized by Sir Arthur Eddington, and so is known as the Eddington factor. In particular, note that for an isotropic radiation field with $I(\mu) = I_o = \text{constant}$, we have

$$f \equiv \frac{K}{J} = \frac{P_R}{U} = \frac{1}{3}. \quad (4.9)$$

This certainly holds very well for the near-isotropic radiation in the stellar interior. But we will see below that the “Eddington approximation”, $f \approx 1/3$, or equivalently $J \approx 3K$, is actually quite useful in modeling the stellar atmosphere as well, even though it ultimately becomes harder to justify near, and especially beyond, the stellar surface.

Exercise: Indeed, show that, far from a stellar surface, the intensity approaches a point source form, $I(\mu) = I_o \delta(\mu - 1)$, for which then $J = H = K$ and thus $f = 1$.

Exercise: On the other hand, consider the physically quite reasonable model that, at the stellar surface, $I(\mu) = I_o$ is constant for $\mu > 1$, and zero otherwise. Show that $f = 1/3$, and thus that the Eddington approximation still holds.

5. Radiation Transfer: Absorption and Emission in a Stellar Atmosphere

The atmospheres and interiors of stars are, of course, not at all a vacuum, and so we don’t expect in general for $I$ to remain spatially constant through a star. Rather the material
Fig. 4.3.— Illustration of the emergent intensity from emission and absorption in a stratified planar atmosphere. The lower boundary at \( z = z_o \) and \( \tau = \tau_o \) is assumed to have an intensity \( I(\mu, \tau_o) \), where \( \mu = \cos \theta \) is the vertical projection cosine of the ray.

Absorption and emission of radiation will in general lead to spatial changes in the radiation field \( I \), something known generally as “radiation transport” or “radiative transfer”. To derive a general equation for radiative transfer, let us first consider the case with just absorption, ignoring for now any emission source of radiation.

5.1. Absorption in Vertically Stratified Planar Layer

This near-exponential stratification of density over a narrow atmospheric layer near the stellar surface implies a strong increase in absorption of any light emitted from lower heights \( z \). The basic situation is illustrated by figure 4.3, but for now neglecting any gas emission \( (\eta = 0) \). For density \( \rho \) and opacity \( \kappa \) over a local height interval \( dz \), the differential reduction
in specific intensity $I(\mu, z)$ along some direction with projection cosine $\mu$ to the vertical gives

$$\frac{dI}{dz} = -\kappa \rho I.$$  \hfill (5.1)

Integration from a lower height $z_o$, with intensity $I(\mu, z_o)$, to a distant observer at $z = +\infty$ gives an observed intensity,

$$I(\mu, z = +\infty) = I(\mu, z_o) e^{-\tau(z_o)/\mu},$$  \hfill (5.2)

where the vertical optical depth from the observer to any height $z$ is defined generally by

$$\tau(z) \equiv \int_z^{\infty} \kappa \rho(z') dz'.$$  \hfill (5.3)

In an atmosphere with a constant opacity $\kappa$ and an exponentially stratified density, $\rho(z) = \rho_* \exp(-z/H)$, this optical depth integral evaluates to

$$\tau(z) = \kappa H \rho_* e^{-z/H} = e^{-z/H}. $$  \hfill (5.4)

where the last equality defines the height $z = 0$ to have a characteristic density $\rho(z = 0) = \rho_* = 1/\kappa H$. Note that this defines this reference height $z = 0$ to have a vertical optical depth $\tau(0) = 1$.

For hot stars with surface temperatures more than about 10,000 K, hydrogen remains almost fully ionized even in the photosphere, and so the opacity near the surface is again roughly set by electron scattering, $\kappa \approx \kappa_e$, while the scale height is again roughly comparable to the solar value $H \approx 300 \text{ km}$. This thus implies a typical photospheric density $\rho_* \approx 10^{-7} \text{ g/cm}^3$.

At the much cooler solar surface, hydrogen is either neutral, or even negatively charged, $H^-$. The latter occurs through an induced dipole binding of a second electron, with ionization potential of just 0.75 eV, i.e. almost a factor 20 lower than the 13.6 eV ionization energy of neutral H. Because a substational fraction of the photons in the photosphere have sufficient energy to overcome this weak binding, it turns out the “bound-free” (b-f) absorption of $H^-$ is a dominant source of opacity in the solar atmosphere, with a characteristic value of $\kappa_{bf} \approx 100 \text{ cm}^2/\text{g}$, i.e. more than a hundred times the opacity from electron scattering. This implies a solar photospheric density is likewise more than a factor hundred lower than in hotter stars, $\rho_* \approx 3 \times 10^{-10} \text{ g/cm}^3$.

5.2. Radiative Transfer Equation for a Planar Stellar Atmosphere

Let us thus now generalize the above pure-absorption analysis to account also for a non-zero radiative emissivity $\eta$, specifying the rate of radiative energy emission per-unit-
volume, and again into some solid angle. As illustrated in figure 4.3, this adds now a positive contribution \( \eta dz/\mu \) to the change in specific intensity \( dI \) defined in eqn. (5.1), yielding a general first-order, ordinary differential equation for the change of intensity with height,

\[
\frac{dI}{dz} = \frac{\eta - \kappa \rho I}{\mu},
\tag{5.5}
\]

Commonly known as the Radiative Transfer Equation (RTE), this represents the basic controlling relation for the radiation field within, and emerging from, a stellar atmosphere. It can alternatively be written with the optical depth \( \tau \) as the independent variable,

\[
\mu \frac{dI}{d\tau} = I - S,
\tag{5.6}
\]

where the ratio of the emission to absorption, \( S \equiv \eta/\kappa \rho \), is called the source function. The emissivity \( \eta \) represents an emitted energy/volume/time/solid-angle, and when this is divided by the opacity and density, with combined units of inverse-length, it gives the source function units of specific intensity (a.k.a. surface brightness).

**Exercise:** Consider a planar slab of physical thickness \( \Delta Z \) and constant density \( \rho \), opacity \( \kappa \), and nonzero emissivity \( \eta \). If the intensity impingent on the slab bottom is \( I_o(\mu) \), compute the emergent \( (\mu > 0) \) intensity \( I(\mu) \) at the slab top, writing this in terms of the slab vertical optical thickness \( \Delta \tau \) and the source function \( S = \eta/\kappa \rho \).

**Solution:** Since the coefficients in eqn. (5.5) are all constant, straightforward integration (using an integrating factor \( \exp[-\kappa \rho z/\mu] \)), gives

\[
I(\mu) = I_o(\mu) e^{-\Delta \tau/\mu} + S \left( 1 - e^{-\Delta \tau/\mu} \right).
\tag{5.7}
\]

In the pure-absorption case \( \eta = S = 0 \), the second (source) term is zero, and we recover the simple result that the lower boundary intensity is just exponentially attenuated. For nonzero \( S \), but no lower boundary intensity, \( I_o = 0 \), we obtain the emergent intensity from slab emission,

\[
I(\mu) = S \left( 1 - e^{-\Delta \tau/\mu} \right) \approx S \quad ; \quad \Delta \tau/\mu \gg 1
\]

\[
\approx S \Delta \tau/\mu = \eta \Delta z/\mu \quad ; \quad \Delta \tau/\mu \ll 1
\]

In cases with significant scattering, the source function can depend in an inherently non-local way on the radiation field itself, and thus can only be solved for in terms of some global model of the atmospheric scattering.
But the problem becomes much simpler in cases like $H^-$ absorption/emission, this source function is just given by the Planck function, $S = B(T)$, which is fixed by the local gas temperature $T(z)$. Indeed, if the opacity, density, and temperature are all known functions of spatial depth $z$, then the temperature, and thus the Planck/Source function, can also be written as a known function of optical depth $\tau$. In this case, the radiative transfer equation, which is just a first-order, ordinary differential equation, can be readily integrated.

### 5.3. The Formal Solution of Radiative Transfer

The resulting integral form of the intensity is commonly called the “Formal Solution”, essentially because it can be ‘formally’ written down even in the case when the source function is non-local, and thus not known as an explicit function of optical depth. Multiplying the transfer equation (5.6) by an integrating factor $e^{-\tau/\mu}$, and then integrating by parts with respect to vertical optical depth $\tau$, we find that for upwardly directed intensity rays with $\mu > 0$, the variation of intensity with optical depth in the planar atmosphere model illustrated in figure 4.3 is given by,

$$
I(\mu, \tau) = I(\mu, \tau_o) e^{(\tau-\tau_o)/\mu} + \int_{\tau_o}^{\tau} S(t) e^{(-t+\tau)/\mu} \frac{dt}{\mu} ; \quad \mu > 0; \quad 0 < \tau < \tau_o, \quad (5.8)
$$

where $\tau_o$ is the total optical thickness of the planar slab under consideration, and $t$ is just a dummy integration variable in optical depth (and not, e.g., the time!). For downward rays with $\mu < 0$, the solution takes the form

$$
I(\mu, \tau) = \int_{0}^{\tau} S(t) e^{(t-\tau)/|\mu|} \frac{dt}{|\mu|} ; \quad \mu < 0; \quad 0 < \tau < \tau_o, \quad (5.9)
$$

where we’ve assumed an upper boundary condition $I(\mu, \tau = 0) = 0$, i.e. no radiation illuminating the atmosphere from above.

### 5.4. Eddington-Barbier Relation for Emergent Intensity

A particularly relevant case for observations of stellar atmospheres is the emergent intensity ($\mu > 0$) from a semi-infinite slab, i.e. for which $\tau_o \to \infty$. The intensity seen by the external observer at $\tau = 0$ is then given as a special case of eqn. (5.8),

$$
I(\mu, 0) = \int_{0}^{\infty} S(t) e^{-t/\mu} \frac{dt}{\mu} ; \quad \mu > 0. \quad (5.10)
$$
To gain insight, let us consider the simple case that the source function is just a linear function of optical depth, \( S(t) = a + bt \). In this case, eqn. (5.10) can be easily integrated analytically (using integration by parts), yielding
\[
I(\mu, 0) = a + b\mu = S(\tau = \mu).
\] (5.11)
Of course the variation of a source function in a stellar atmosphere is usually more complicated than this linear function, but the approximate result,
\[
I(\mu, 0) \approx S(\tau = \mu),
\] (5.12)
which is known as the Eddington-Barbier (E-B) relation, turns out to be surprisingly accurate for a wide range of conditions. The reason is that the exponential attenuation represents a strong localization of the integrand in eqn. (5.10), meaning then that a first-order Taylor expansion gives a roughly linear variation of the source function around the region where the optical depth terms in the exponential are of order unity.

5.5. Limb Darkening of Solar Disk

A key application of the E-B relation regards the variation of the sun’s surface brightness as one looks from the center to limb of the solar disk. At disk center, one is looking vertically

![Visible light picture of the solar disk, showing the center to limb darkening of the surface brightness.](image-url)
down into the local planar atmosphere, i.e. with $\mu = 1$. On the limb, one’s view just grazes the atmosphere nearly along the local horizontal, i.e. with $\mu = 0$. The E-B relation gives for the limb/center brightness ratio,

$$\frac{I(0,0)}{I(1,0)} = \frac{a}{a+b} = \frac{S(0)}{S(0) + dS/d\tau} \approx \frac{B(0)}{B(0) + dB/d\tau}.$$  

(5.13)

The second equality uses the fact that the linear source function coefficients $a$ and $b$ just represent the surface value and gradient of the source function, $a = S(0)$ and $b = S' = dS/d\tau$. The last approximation assumes the LTE case that $S = B$. Since the Planck function depends only on temperature, $B = B(T)$, we have

$$\frac{dB}{d\tau} = -\frac{dT}{dz} \frac{1}{\kappa \rho} \frac{dB}{dT}.$$  

(5.14)

Since $dB/dT > 0$, and since the temperature of an atmosphere generally declines with height, i.e. $dT/dz < 0$, we see that $dB/d\tau > 0$.

Eqn. (5.13) thus predicts $I(0,0)/I(0,1) < 1$, and so an overall limb darkening of the solar surface brightness. As illustrated by the image of the solar disk in figure 5.1, this is indeed what is observed.

Measurements of the functional variation of solar brightness across the solar disk can even be used to infer the temperature structure of the solar atmosphere. We will consider this further later when we develop solar atmosphere models.

**Exercise:** For far UV wavelengths $\lambda < 912\text{Å}$, the photon energy $E > 13.6$ eV is sufficient to ionize neutral hydrogen, and so the associated bound-free absorption by hydrogen makes the opacity in the far UV very high.

a. Compared to visible light with lower opacity, are we able to see to deeper or shallower heights in the far UV?

b. If a far UV picture of the sun shows limb brightening instead of limb darkening, what does that suggest about the temperature stratification of the solar atmosphere?

c. Combining this with results for the limb darkening in optical light, sketch the overall variation of the sun’s temperature $T$ vs. height $z$.

### 6. Emission, absorption, and scattering: LTE vs. NLTE

In addition to absorption, the material in a stellar atmosphere (or indeed most any matter with a finite temperature) will generally also emit radiation, for example through the
inverse processes to absorption. Since microscopic atomic processes in physics are generally all symmetric under time reversal, this can be thought of as somewhat like running the clock backward.

For example, for the sun a dominant source of emission is from the “free-bound” (f-b) recombination of electrons and neutral hydrogen to form the $H^-$, reversing then the “bound-free” (b-f) absorption that dissociates the $H^-$ ion.

Superficially, the overall process may appear to resemble electron scattering, with absorption of radiation by one $H^-$ ion followed shortly thereafter by remission of radiation during the formation of another $H^-$ ion.

But a key distinction is that, in constrast to the conservative nature of scattering – wherein the energy of the scattered photon essentially remains the same –, the sequence of $H^-$ absorption and emission involves a constant shuffling of the photon energy, very effectively coupling it to the pool of thermal energy in the gas, as set by the local temperature $T(z)$. This is very much the kind of process that tends to quickly bring the radiation and gas close to a Local Thermodynamic Equilibrium (LTE). As discussed in §4.2 of DocOnotes-stars1, this means that the local radiation field becomes quite well described by the Planck Blackbody function given in eqns. (4.3) and (4.4).

In contrast, because the opacity of hotter stars is dominated by electron scattering, with thus much weaker thermal coupling between the gas and radiation, the radiation field and emergent spectrum can often deviate quite markedly from what would be expected in LTE. Instead one must develop much more complex and difficult non-LTE (NLTE) models for such hot stars. We will later outline the procedures for such NLTE models, including also some specific features of the solar spectrum (e.g. the so-called “H and K” lines of Calcium) that also require an NLTE treatment.

### 6.1. Absorption and Thermal Emission: LTE with $S = B$

One important corollary is that processes in LTE exhibit a “detailed balance”, i.e. a direct link between each process and its inverse\(^6\). Thus for example, the thermal emissivity

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\(^5\)That is, ignoring the electron recoil that really only becomes significant for gamma-ray photons with energies near or above the electron rest mass energy, $m_ec^2 \approx 0.5$ MeV.

\(^6\)Einstein exploited this to write a set of relations between the atomic coefficients (related to cross section) for absorption and their emission inverse. As such, once experiments or theoretical computations give one, the inverse is also directly available. See §9.1 below.
\( \eta_{th} \) from LTE processes like \( H^- \) f-b emission is given by the associated bound-free absorption opacity \( \kappa_{abs} \) times the local Planck function,

\[
\eta_{th} = \kappa_{abs} \rho B(T) .
\] (6.1)

This implies then that, for such pure-absorption and thermal emission, we obtain the simple LTE result,

\[
S(\tau) = \frac{\eta_{th}}{\kappa_{abs} \rho} = B(T(\tau)) .
\] (6.2)

### 6.2. Pure Scattering Source Function: \( S = J \)

In contrast, for the case of pure scattering opacity \( \kappa_{sc} \), the associated emission becomes completely insensitive to the thermal properties of the gas, and instead depends only on the local radiation field. If we assume (or approximate) the scattering to be roughly isotropic, the scattering emissivity \( \eta_{sc} \) in any direction depends on both the opacity and the angle-averaged mean-intensity,

\[
\eta_{sc} = \kappa_{sc} \rho J .
\] (6.3)

This implies then that, for pure-scattering,

\[
S(\tau) = J(\tau) .
\] (6.4)

Since \( J \) is the angle-average of \( I \), which itself depends on the global integral given by the formal solution (eqns. 5.8 and 5.9), the solution is inherently non-local, representing then a case of non-LTE or NLTE.

### 6.3. Source Function for Scattering and Absorption: \( S = \epsilon B + (1 - \epsilon)J \)

For the general case in which the total opacity consists of both scattering and absorption, \( k \equiv \kappa_{abs} + \kappa_{sc} \), the total emissivity likewise contains both thermal and scattering components

\[
\eta = \eta_{th} + \eta_{sc} = \kappa_{abs} \rho B + \kappa_{sc} \rho J .
\] (6.5)

If we then define an absorption fraction

\[
\epsilon \equiv \frac{\kappa_{abs}}{\kappa_{abs} + \kappa_{sc}} ,
\] (6.6)

we can write the general source function in the form

\[
S = \epsilon B + (1 - \epsilon)J .
\] (6.7)
6.4. Thermalization Depth vs. Optical Depth

In physical terms, the absorption fraction $\epsilon$ can be also thought of as a photon destruction probability per encounter with matter. In cases when $\epsilon \ll 1$, a photon that is created thermally somewhere within an atmosphere can scatter many ($N_{sc} \approx 1/\epsilon$) times before it is likely to be absorbed. By a simple random walk argument, the root-mean-square number of mean-free-paths between its thermal creation and absorptive destruction is thus $\sqrt{N} = 1/\sqrt{\epsilon}$, which thus corresponds to an optical depth change of $\Delta \tau = 1/\sqrt{\epsilon}$. This implies that from locations of the atmosphere with total vertical optical depth $\tau < \Delta \tau = 1/\sqrt{\epsilon}$, any photon that is thermally created will generally escape the star, instead of being destroyed by a true absorption.

The thermalization depth,

$$\tau_{th} = \frac{1}{\sqrt{\epsilon}},$$  \hspace{1cm} (6.8)

thus represents the maximum optical depth from which thermally created photons can scatter their way to escape through the stellar surface, without being destroyed by a true absorption.

It is important to understand the distinction here between thermalization depth and optical depth. If we look vertically into a stellar atmosphere, we can say that the photons we see had their last encounter with matter at an optical depth of order unity, $\tau \approx 1$.

But the energy that created that photon can, in the strong scattering case with $\epsilon \ll 1$, come typically from a much deeper layer, characterized by the thermalization depth $\tau_{th} \approx 1/\epsilon \gg 1$.

A physical interpretation of the Eddington-Barbier relation comes from the first notion that we can see vertically down to a layer of optical depth order unity, and so the observed vertical intensity just reflects the source function at that layer, $I(1,0) \approx S(\tau = 1)$.

But in cases with strong scattering, any thermal emission within a thermalization depth of the surface, $\tau < \tau_{th}$ can escape to free space. Since this represents a loss or “sink” of thermal energy, the source function in this layer general becomes reduced relative to the local Planck function, i.e.

$$S(\tau < \tau_{th}) < B(\tau).$$  \hspace{1cm} (6.9)

In particular, since $S(\tau = \mu) < B(\tau = \mu)$, one can no longer directly infer $B(\tau = \mu)$, or the associated surface temperature temperature $T(\tau = \mu)$, by applying the Eddington-Barbier relation to interpretation of observations of the emergent intensity $I(\mu, 0)$.

In general, we must thus go down to deep layer with $\tau \gtrsim \tau_{th}$ to recover the LTE
condition,

\[ S(\tau \gtrsim \tau_{th}) \approx B(\tau). \]  

(6.10)

For example, in the atmospheres of hot stars – for which the opacity is dominated by free-electron scattering – the photon destruction probability can be quite small, e.g. \( \epsilon \approx 10^{-4} \), implying that LTE is only recovered at depths \( \tau \gtrsim 100 \).

But note in the case of nearly pure-absorption – which is not a bad approximation for the solar atmosphere wherein the opacity is dominated by \( H^{-} \) b-f absorption – we do recover the LTE result quite near the visible surface, \( \tau \approx \tau_{th} \approx 1 \). So solar limb darkening can indeed be used to infer the temperature stratification of the solar atmosphere.

### 6.5. Effectively Thick vs. Effectively Thin

Associated with the thermalization depth is the concept of effective thickness, to be distinguished from optical thickness.

A planar layer of material with total vertical optical thickness \( \tau \) is said to be optically thin if \( \tau < 1 \), and optically thick if \( \tau > 1 \).

But it is only effectively thick if \( \tau > \tau_{th} \). If \( \tau < \tau_{th} \), it is effectively thin, even in cases when it is optically thick, i.e. with \( 1 < \tau < \tau_{th} \).

We will discuss below solutions of the full radiative transfer for planar slabs that are effectively thick vs. thin.

### 7. Properties of the Radiation Field

#### 7.1. Moments of the Transfer Equation

The radiation field moments \( J, H, \) and \( K \) defined above provide a useful way to characterize key properties like energy density, flux, and radiation pressure, instead of dealing with the more complete angle dependence given by the full specific intensity \( I(\mu) \). To relate these radiation moments to their physical source and dependence on optical depth, it is convenient to carry out progressive angle moments \( j = 0, 1, \ldots \) of the radiative transfer equation itself,

\[
\frac{1}{2} \int_{-1}^{+1} d\mu \, \mu^j \frac{dI}{d\tau} = \frac{1}{2} \int_{-1}^{+1} d\mu \, \mu^j (I - S). \]  

(7.1)
Since optical depth is independent of $\mu$, we can pull the $d/d\tau$ operator outside the integral. The $j = 0$ or “0th” moment equation then becomes

$$\frac{dH}{d\tau} = J - S. \quad (7.2)$$

Note for example that in the case of pure, conservative scattering we have $S = J$, implying $dH/d\tau = 0$ and thus a spatially constant flux $H$. Since scattering neither creates nor destroys radiation, but simply deflects its direction, the flux in a scattering atmosphere must be everywhere the same constant value.

For the first ($j = 1$) moment of the transfer equation, the oddness of the integrand over the (angle-independent) source function $S$ means that the contribution of this term vanishes, leaving

$$\frac{dK}{d\tau} = H. \quad (7.3)$$

In both eqns. (7.2) and (7.3), note that a lower moment on the right-hand-side, e.g. $J$ or $H$, is related to the optical depth derivative of a higher moment, $H$ or $K$, on the left-hand-side. In principal, one can continue to define higher moments, but both the usefulness and physical interpretation become increasingly problematic.

To truncate the process, we need a closure relation that relates a higher moment like $K$ to a lower moment like $J$. As noted previously, a particular useful example is the Eddington approximation, $J \approx 3K$, which then implies

$$\frac{1}{3} \frac{dJ}{d\tau} \approx H, \quad (7.4)$$

which when combined with the zeroth moment eqn. (7.2), yields a 2nd order ODE for $J$,

$$\frac{1}{3} \frac{d^2J}{d\tau^2} = J - S. \quad (7.5)$$

Given $S(\tau)$, this can be readily integrated to give $J(\tau)$.

### 7.2. Diffusion Approximation at Depth

At sufficiently deep layers of the atmosphere, i.e. with large optical depths beyond the thermalization depth, $\tau \gg \tau_{th}$, we expect the radiation field to become isotropic and near the local Planck function, $J \rightarrow S \rightarrow B$. Let us thus assume that the variation of the Source function near some reference depth $\tau$ can be written as a Taylor expansion of the Planck function about this depth,

$$S(t) \approx B(\tau) + \left. \frac{dB}{d\tau} \right|_\tau (t - \tau) + O \left( \frac{d^2B}{d\tau^2} \right), \quad (7.6)$$
where, noting that each higher term is Order $1/\tau$ (commonly written $O(1/\tau)$) smaller than the previous, we truncate the expansion after just the second, linear term. Application in the formal solution (5.8) then gives for the local intensity,

$$I(\mu, \tau) \approx B(\tau) + \mu \frac{dB}{d\tau}.$$  \hspace{1cm} \text{(7.7)}

Applying this in the definitions of the radiation field moments gives

$$J(\tau) \approx B(\tau) + O\left(\frac{d^2 B}{d\tau^2}\right)$$  \hspace{1cm} \text{(7.8)}

$$H(\tau) \approx \frac{1}{3} \frac{dB}{d\tau} + O\left(\frac{d^3 B}{d\tau^3}\right)$$  \hspace{1cm} \text{(7.9)}

$$K(\tau) \approx \frac{1}{3} B(\tau) + O\left(\frac{d^2 B}{d\tau^2}\right).$$  \hspace{1cm} \text{(7.10)}

If we keep just the first-order terms, then comparison of eqns. (7.8) and (7.10) immediately recovers the Eddington approximation, $J = 3K$, while the flux $H$ is given by the diffusion approximation form,

$$H \approx \frac{1}{3} \frac{dB}{d\tau} = -\left[\frac{1}{3\kappa\rho} \frac{\partial B}{\partial T}\right] \frac{dT}{dz},$$  \hspace{1cm} \text{(7.11)}

where the latter equality shows how this diffusive flux scales directly with the vertical temperature gradient $dT/dz$, much as it does in, e.g., conduction. Indeed, the terms in square brackets can be thought of as a radiative conductivity, which we note depends inversely on opacity and density, $1/\kappa\rho$.

7.3. The Rosseland Mean Opacity for Diffusion of Total Radiative Flux

Note that the physical radiative flux is just $F = 4\pi H$. In modeling stellar interiors, we will use this corresponding flux form in spherical symmetry, replacing height with radius, $z \rightarrow r$,

$$F_\nu(r) \approx -\left[\frac{4\pi}{3\kappa\rho} \frac{\partial B_\nu}{\partial T}\right] \frac{dT}{dr},$$  \hspace{1cm} \text{(7.12)}

where we now have also reintroduced subscripts $\nu$ to emphasize that this, like all the equations above, depends in principal on photon frequency.

\footnote{For a lucid summary of radiative transfer in stellar interiors, see Rich Townsend’s notes 06radiation.pdf.}
But to model the overall energy transport in a stellar atmosphere or interior, we need the total, frequency-integrated (a.k.a. bolometric) flux

$$F(r) \equiv \int_{0}^{\infty} F_{\nu} d\nu.$$  \hfill (7.13)

Collecting together all of the frequency-dependent terms of $F_{\nu}$, we have

$$F(r) = -\frac{4\pi}{3\rho} \frac{dT}{d\nu} \int_{0}^{\infty} \frac{1}{\kappa_{\nu}} \frac{\partial B_{\nu}}{\partial T} d\nu.$$  \hfill (7.14)

The required frequency integral can be conveniently represented by introducing the Rosseland mean opacity, defined by

$$\kappa_{R} \equiv \frac{\int_{0}^{\infty} \frac{\partial B_{\nu}}{\partial T} d\nu}{\int_{0}^{\infty} \frac{1}{\kappa_{\nu}} \frac{\partial B_{\nu}}{\partial T} d\nu}.$$  \hfill (7.15)

We can see that $\kappa_{R}$ represents a harmonic (i.e. inverse) mean of the frequency-dependent opacity $\kappa_{\nu}$, with $\partial B_{\nu}/\partial T$ as a weighting function. The numerator can be readily evaluated by taking the temperature derivative outside the frequency integral,

$$\int_{0}^{\infty} \frac{\partial B_{\nu}}{\partial T} d\nu = \frac{\partial}{\partial T} \int_{0}^{\infty} B_{\nu} d\nu = \frac{\partial}{\partial T} \left( \frac{\sigma T^{4}}{\pi} \right) = \frac{4\sigma T^{3}}{\pi},$$ \hfill (7.16)

where the Stefan-Boltzmann constant $\sigma$ is given by

$$\sigma = \frac{2\pi^{5} k^{4}}{15 c^{2} \hbar^{3}}.$$  \hfill (7.17)

We can thus write the integrated flux as

$$F(r) = -\left[ \frac{16\sigma}{3} \frac{T^{3}}{\kappa_{R}\rho} \right] \frac{dT}{dr}.$$  \hfill (7.18)

This final result – which tells us the total radiative flux $F$ given the temperature, its gradient, the density and the Rosseland-mean opacity – is known as the radiative diffusion equation. Sometimes it is instructive to write this as

$$F(r) = -c \frac{1}{3\kappa_{R}\rho} \frac{dU}{dr},$$ \hfill (7.19)

where

$$U \equiv 4\sigma T^{4}/c.$$  \hfill (7.20)

is the energy density of radiation, with CGS units erg/cm$^{3}$. Alternatively, in terms of the radiation pressure $P_{rad} = U/3$, we can write

$$\frac{dP_{rad}}{d\tau_{R}} = \frac{F}{c},$$ \hfill (7.21)

wherein we have also now cast the radial variation in terms of the Rosseland optical depth $d\tau_{R} = \kappa_{R}\rho dr$. 

7.4. Exponential Integral Moments of Formal Solution: the Lambda Operator

Let us now return to the problem of solving for the radiation moments in the full atmosphere where the above diffusion treatment can break down. Applying the definition of mean intensity $J$ from eqn. (4.2) into the formal solution eqns. (5.8) and (5.9), we find for the case of semi-infinite atmosphere (i.e., allowing the lower boundary optical depth $\tau_o \to \infty$; and again suppressing the $\nu$ subscripts for simplicity),

$$J(\tau) = \frac{1}{2} \int_0^1 d\mu \int_{\tau}^{\infty} S(t) e^{- (t-\tau)/\mu} \frac{dt}{\mu} + \frac{1}{2} \int_{-1}^0 d\mu \int_{0}^{\tau} S(t) e^{- (\tau-t)/|\mu|} \frac{dt}{|\mu|}$$

(7.22)

$$= \frac{1}{2} \int_0^{\infty} S(t) E_1 (|t-\tau|) \frac{dt}{\mu}$$

(7.23)

$$\equiv \Lambda_\tau [S(t)].$$

(7.24)

Here the second equality uses the first ($n = 1$) of the general exponential integral defined by,

$$E_n(x) \equiv \int_1^{\infty} e^{-xt} \frac{dt}{t^n}. \quad (7.25)$$

Some simple exercises in the homework problem set help to illustrate the general properties of exponential integrals. An essential point is that they retain the strong geometric factor attenuation with optical depth, and so can be qualitatively thought of just a fancier form of the regular exponential $e^{-\tau}$.

**Exercise:** Given the definition of the exponential integral in eqn. (7.25), prove the following properties:

a. $E'_n(x) = -E_{n-1}(x)$.

b. $E_n(x) = [e^{-x} - xE_{n-1}(x)]/(n - 1)$.

c. $E_n(0) = 1/(n - 1)$

d. $E_n(x) \approx e^{-x}/x$ for $x \gg 1$.

The last equality in (7.24) defines the *Lambda Operator* $\Lambda_\tau [S(t)]$, which acts on the full source function $S[t]$. In the general case in which scattering gives the source function a dependence on the radiation field $J$, finding solutions for $J$ amounts to solving the operator equation,

$$J(\tau) = \Lambda_\tau [\epsilon B(t) + (1 - \epsilon)J(t)],$$

(7.26)

which states that the *local* value of intensity at any optical depth $\tau$ depends on the *global* variation of $J(t)$ and $B(t)$ over the whole atmosphere $0 < t < \infty$. One potential approach to
solving this equation is to simply assume some guess for $J(t)$, along with a given $B(t)$ from the temperature variation $T(t)$, then compute a new value of $J(\tau)$ for all $\tau$, apply this new $J$ into the Lambda operator, and repeat the whole process it converges on a self-consistent solution for $J$.

Unfortunately, such *Lambda iteration* turns out to have a hopelessly slow convergence, essentially because the ability of deeper layers to influence upper layers scales as $e^{-\tau}$ with optical depth, implying it can take $e^\tau$ steps to settle to a full physical exchange that ensures full convergence. However, a modified form known as *Accelerated Lambda Iteration* (ALI) turns out to have a suitably fast and stable convergence, and so is often used in solving radiative transfer problems in stellar atmospheres. But this is rather beyond the scope of the summary discussion here.

One can likewise define a formal solution integral for the flux,

$$H(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} S(t) E_2(t - \tau) \, dt - \frac{1}{2} \int_{0}^{\tau} S(t) E_2(\tau - t) \, dt,$$

which can be used to define another integral operator, commonly notated $\Phi$. An analogous equation can be written for the K-moment, which involves $E_3$, and can be used to define yet another operator, commonly denoted $X$.

**Exercise:** Assume a source function that varies linearly with optical depth, i.e. $S(t) = a + bt$, where $a$ and $b$ are constants.

a. Apply this $S(t)$ in eqn. (7.24) and carry out the integration to obtain the optical depth variation of mean intensity $J(\tau)$ in terms of exponential integrals $E_n(\tau)$.

b. Next apply $S(t)$ in eqn. (7.27) and carry out the integration to obtain the optical depth variation of the Eddington flux $H(\tau)$ in terms of exponential integrals $E_n(\tau)$.

c. For the case $a = 2$ and $b = 3$, plot $H(\tau)$ and the ratio $J(\tau)/S(\tau)$ vs. $\tau$ over the range $[0, 2]$.

### 7.5. The Eddington-Barbier Relation for Emergent Flux

For the case of a source function that is linear in optical depth, $S(t) = a + bt$, we can use eqn. (7.27) to derive a formal solution for the emergent flux,

$$H(0) = \frac{1}{2} \int_{0}^{\infty} (a + bt) E_2[t] \, dt = \frac{a}{4} + \frac{b}{6}. \quad (7.28)$$
Recalling that the physical flux $F = 4\pi H$, we thus find that

$$F(0) = \pi (a + (2/3)b) = \pi S(\tau = 2/3),$$

which now represents a form of the Eddington-Barbier relation for the emergent flux, associating this with the source function at a characteristic, order-unity optical depth $\tau = 2/3$. This can be used to interpret the observed flux from stars for which, unlike for the sun, we cannot resolve the surface brightness, $I(\mu, 0)$. Indeed, note that the basic Eddington flux relation (7.28) can also be derived by taking the outward flux moment of the Eddington-Barbier relation for emergent intensity (5.11).

### 7.6. Radiative Equilibrium

Since atmospheric layers are far away from the nuclear energy generation of the stellar core, there is no net energy produced in any given volume. If we further assume energy is transported fully by radiation (i.e. that conduction and convection are unimportant), then the total, frequency-integrated radiative energy emitted in a volume must equal the total radiative energy absorbed in the same volume. This condition of radiative equilibrium can be represented in various alternative forms,

$$\int_0^\infty \int_0^{2\pi} d\Omega \, \eta_\nu = \int_0^\infty \int_0^{2\pi} d\Omega \, \rho k_\nu I_\nu,$$

$$\int_0^\infty k_\nu S_\nu \, d\nu = \int_0^\infty k_\nu J_\nu \, d\nu,$$

$$\int_0^\infty \kappa_\nu B_\nu \, d\nu = \int_0^\infty \kappa_\nu J_\nu \, d\nu,$$

where $k_\nu = \kappa_\nu + \sigma_\nu$ is the total opacity at frequency $\nu$, with contributions from both absorption ($\kappa_\nu$) and scattering ($\sigma_\nu$) opacity components. The second relation uses the isotropy and other basic properties of the volume emissivity,

$$\eta_\nu = k_\nu S_\nu = \kappa_\nu B_\nu + \sigma_\nu J_\nu.$$

The third relation uses the conservative property of the coherent scattering component at each frequency $J_\nu = 4\pi \eta_\nu^c / \sigma_\nu$, and shows that, when integrated over frequency, the total true absorption of radiation must be balanced by the total thermal emission.

Upon frequency integration of the flux moment of the radiative transfer equation, we find for the bolometric fluxes $H$ or $F$

$$\frac{dH}{dz} = \frac{dF}{dz} = 0,$$
which shows that the bolometric flux is spatially constant throughout a planar atmosphere in radiative equilibrium.

Indeed, along with the surface gravity, the impingent radiative flux $F$ from the underlying star represents a key characteristic of a stellar atmosphere. Moreover, since the flux from a blackbody defines an effective temperature $T_{\text{eff}}$ through $F = \sigma T_{\text{eff}}^4$, models of planar stellar atmospheres are often characterized in terms of just the two parameters: log $g$ and $T_{\text{eff}}$. By contrast, the overall structure of a star is generally described by three parameters, e.g. luminosity $L$, mass $M$, and radius $R$. But because the radius cancels in the ratio of surface gravity to surface flux, just log $g$ and $T_{\text{eff}}$ suffice to characterize a planar atmosphere, whose properties are insensitive to the stellar radius $R$.

### 7.7. Two-Stream Approximation for Radiative Transfer

An alternative to using moment equations is to approximate the radiative field as confined to just two rays, one upward (+) and the other downward (-), with associated intensities $I^+$ and $I^-$. In this approach, one has the freedom to choose the magnitude of the projection cosine of these rays, $|\mu| = \mu_1$. In this two-stream model the transfer equation for each of the \pm directions can be written

$$\pm \mu_1 \frac{dI^\pm}{d\tau} = I^\pm - S.$$  \hfill (7.35)

The mean intensity and Eddington flux now reduce from integrals to mere sums,

$$J = \frac{I^+ + I^-}{2},$$  \hfill (7.36)

and

$$H = \mu_1 \frac{I^+ - I^-}{2}.$$  \hfill (7.37)

Adding the (+) and (-) transfer equations then gives

$$\frac{dH}{d\tau} = J - S = \epsilon(J - B),$$  \hfill (7.38)

where the latter equality again assumes a source function of the form $S = (1 - \epsilon) J + \epsilon B$.

Subtracting the two equations (7.35) gives

$$\mu_1 \frac{dJ}{d\tau} = \frac{H}{\mu_1}.$$  \hfill (7.39)

Combining these gives a second order equation in $J$,

$$\mu_1^2 \frac{d^2 J}{d\tau^2} = J - S = \epsilon(J - B).$$  \hfill (7.40)
Comparison with eqn. (7.5) shows that, if we set $\mu_1 = 1/\sqrt{3}$, then this two-stream approach becomes fully equivalent to moment approach that assumes the Eddington approximation.

### 7.8. Transmission through Planar Layer: Scattering vs. Absorption

We can use this two-stream approximation to estimate the transmission through a pure-scattering layer or cloud, much like the clouds here on earth. For such a pure scattering ($\epsilon = 0$) case, the right side of eqn. (7.38) is zero, implying then that the flux $H$ must be constant; physically, this again reflects the fact that for pure-scattering, radiation can be neither created or destroyed. By integration of eqn. (7.39), we find that the mean intensity must vary linearly with optical depth,

$$J(\tau) = \frac{H \tau}{\mu_1^2} + C,$$

(7.41)

where $C$ is an integration constant. This can be evaluated by boundary conditions on the $I^\pm$ at the cloud’s top and bottom.

For a cloud in which the vertical optical depth ranges from $\tau = 0$ at the top to some total cloud optical thickness $\tau_c$ at the bottom, let us assume an incoming (downward) intensity $I^-(0) = I_o$ impinging along the downward direction $-\mu_1$ at the top, with zero upward intensity at the bottom, $I^+(\tau_c) = 0$. Note that

$$I^\pm(\tau) = J \pm \frac{H}{\mu_1} \left( \frac{\tau}{\mu_1} \pm 1 \right) + C$$

(7.42)

$$= \frac{H}{\mu_1} \left( \frac{\tau}{\mu_1} \pm 1 + 1 \right) + I_o,$$

(7.43)

(7.44)

where the third equality follows from the upper boundary condition $I^-(0) = I_o$. Applying the lower boundary condition $I^+(\tau_c) = 0$ then gives for the (downward) transmitted flux through the cloud

$$H = \frac{\mu_1 I_o}{\tau_c/\mu_1 + 2} = \frac{H_o}{\tau_c/2 + 1/\mu_1},$$

(7.45)

where the latter equality scales the transmitted flux by its value $H_o$ without any cloud layer, i.e., with $\tau_c = 0$. If we assume nearly vertical illumination $-\mu_1 \approx -1$ (as would apply near noon around midsummer), then we see that a scattering cloud’s reduction of the sun’s flux scales as $1/\left(\tau_c/2 + 1\right)$. A typically very cloudy day with $\tau_c \approx 10$ would thus have the sun’s
flux reduced by a factor 1/6. Thus when taking a picture, you might need to increase the exposure time by a factor of 6.

This relatively modest reduction should be contrasted with the much stronger, exponential attenuation of the transmitted flux by a pure-absorbing cloud with the same optical thickness,

$$H_{\text{abs}} = -(\mu_1 I_o/2)e^{-\tau_c/\mu_1} = H_o e^{-\tau_c/\mu_1}.$$  \hspace{1cm} (7.46)

For the same example with $\mu_1 = 1$ and $\tau_c = 10$, this would give a much stronger flux reduction of about a factor $e^{-10} \approx 5 \times 10^{-5}$. Thus, for example, if the clouds were made of black absorbing coal dust instead of highly reflective water vapor, a cloudy day would be nearly pitch dark! Thus dust from a volcano, or from the meteor impact that killed the dinosaurs, could make the affected surface quite dark, and cold.

This difference in flux attenuation is one of the key physical distinctions between a scattering vs. absorption layer.

8. Radiative Transfer for Gray Opacity

In general, the complex frequency dependence of stellar opacity greatly complicates the full solution for the radiation field. But we can gain great insight into the overall properties of an atmosphere and its radiation if we make the (strong) simplifying assumption that the opacity is gray, i.e. independent of frequency or wavelength. In this case the Rosseland mean opacity is just given by this constant, gray opacity, $\kappa_R = \kappa$. If we now identify unsubscripted symbols for the intensity $I$ and its angle moments $J$, $H$, and $K$ with their bolometric values, then the transfer equation and its moments have the basic forms defined above,

$$\mu \frac{dI}{d\tau} = I - S \hspace{1cm} (8.1)$$

$$\frac{dH}{d\tau} = J - S \hspace{1cm} (8.2)$$

$$\frac{dK}{d\tau} = H. \hspace{1cm} (8.3)$$


Now if we further assume a condition of radiative equilibrium, we have generally $S = J$, regardless of the admixture of absorption and thermal emission vs. scattering. As such, the
second equality above immediately implies a constant flux $H$. This allows us to immediately integrate to obtain for the $K$-moment

$$K(\tau) = H(\tau + C), \quad (8.4)$$

where $C$ is an integration constant. A complete solution requires now that we relate $K$ to $J$. So recall again that at large depth $\tau \gg 1$ we recover a diffusion limit for which the radiation field has only a small deviation from isotropy, giving then the Eddington approximation $J \approx 3K$. This suggests we write the solution for mean intensity in the form,

$$J(\tau) = 3H(\tau + q(\tau)), \quad (8.5)$$

where $q(\tau)$ is called the “Hopf function”, and use of the Eddington approximation at large optical depth $\tau \to \infty$ shows that $q(\infty) = C$. Application of $S = J$ from eqn. (8.5) in the formal solution for mean intensity (7.24) means that finding the Hopf function requires solving the integral equation,

$$\tau + q(\tau) = \frac{1}{2} \int_0^\infty (t + q(t)) \, E_1[|t - \tau|] \, dt. \quad (8.6)$$

### 8.2. The Eddington Gray Atmosphere

While it is possible to tabulate full solutions of the integral equation (8.6), a more analytically tractable approach is to assume validity of the Eddington approximation everywhere. Recalling from the exercises that the Eddington approximation nearly holds for a wide range of forms for $I(\mu)$, such an Eddington gray atmosphere approach, while not exact, turns out to give pretty accurate, and very insightful, results.

Using $J = 3K$, let us now rewrite eqn. (8.4) as a solution for the Eddington approximation for mean intensity

$$J_E(\tau) = 3H\tau + C', \quad (8.7)$$

where $C' = 3HC$ is just an alternative definition for the integration constant. Using the formal solution eqn. (7.27), we find that the surface flux is

$$H(0) = \frac{1}{2} \int_0^\infty J(t)E_2(t) \, dt \quad (8.8)$$

$$= \frac{1}{2} \int_0^\infty (3H\tau + C') \, E_2(t) \, dt \quad (8.9)$$

$$= C' \frac{H}{4} + \frac{H}{2}. \quad (8.10)$$
Setting $H(0) = H$, we find $C' = 2H$, implying $C = 2/3$, and thus that for the Eddington approximation, the Hopf function is just a constant, $q_E(\tau) = 2/3$. This gives a simple analytic form for the mean intensity in an Eddington gray atmosphere,

$$J_E(\tau) = 3H \left( \tau + \frac{2}{3} \right). \quad (8.11)$$

Using then $J_E = B(T) = \sigma T^4/\pi$ and $H = \sigma T_{\text{eff}}^4/4\pi$, we find that the temperature in a gray atmosphere varies according to

$$T^4(\tau) = \frac{3}{4} T_{\text{eff}}^4 \left( \tau + \frac{2}{3} \right). \quad (8.12)$$

This shows that the temperature increases in the inner regions with large optical depth, giving $T \approx T_{\text{eff}} \tau^{1/4}$ for $\tau \gg 1$. Note moreover that $T(\tau = 2/3) = T_{\text{eff}}$, showing again that such $\tau$ of order unity corresponds to roughly to the visible photosphere. On the other hand, for very small optical depth, we find a “surface temperature” $T_0 \equiv T(0) = T_{\text{eff}}/2^{1/4} \approx 0.841 T_{\text{eff}}$. This agrees pretty closely with the exact value for a non-Eddington gray model $T_0/T_{\text{eff}} = (\sqrt{3}/4)^{1/4} \approx 0.8114$.

### 8.3. Eddington Limb-Darkening Law

Let us next apply this Eddington gray atmosphere result into the formal solution (5.10) for the surface intensity,

$$I_E(\mu, 0) = 3H \int_0^\infty (t + 2/3) e^{-t/\mu} dt/\mu \quad (8.13)$$

$$= 3H(\mu + 2/3). \quad (8.14)$$

This gives the Eddington limb-darkening law

$$\frac{I_E(\mu, 0)}{I_E(1, 0)} = \frac{3}{5} \left( \mu + \frac{2}{3} \right). \quad (8.15)$$

For example, this predicts a limb-to-center brightness ratio $I_E(0, 0)/I_E(1, 0) = 2/5 = 0.4$, which is in good agreement with optical observations of the solar disk.

### 8.4. Lambda Iteration of Eddington Gray Atmosphere

It is important to realize that, while very helpful for providing insight, this Eddington gray atmosphere model is not a fully self-consistent solution for the radiation transport.
To see this, let us compute a new intensity from the Lambda operator form of the formal solution (7.24),

\[
J^{(1)}_E(\tau) = \Lambda_\tau \left[ J^0_E(t) \right] \\
= 3H \Lambda_\tau \left[ t + 2/3 \right] \\
= 3H \left( \tau + 2/3 + E_3(\tau)/2 - E_2(\tau)/3 \right),
\]

where the superscript indicates that this is a first-order iteration on the basic Eddington solution, \(J^0_E\). For large optical depth \(\tau \gg 1\) the exponential integral terms all vanish, and so we find \(J^{(1)}_E \to J^0_E\), showing that this Lambda iteration does not affect the solution deep in the atmosphere, where the Eddington approximation is indeed well justified. But at the surface we find \(J^{(1)}_E(0)/J^0_E(0) = 7/8\), so the solution has changed by \(1/8\), or 12%. This now gives a surface temperature \(T_o/T_{\text{eff}} = (7/16)^{1/4} = 0.813\), which is substantially closer to the exact result \(T_o/T_{\text{eff}} = 0.8114\) than the earlier result \(T_o/T_{\text{eff}} = 1/2^{1/4} = 0.841\).

As shown in the exercise below, a similar application of the Eddington solution for mean intensity into the formal solution for the flux shows that the flux is not constant, as required by radiative equilibrium. This thus represents an inherent *inconsistency* in the Eddington gray atmosphere model; then again, as shown in the exercise, the relative error is quite small, at most only a few percent.

**Exercise:**

a. Apply \(S(t) = J_E(t)\) in the formal integral solution for the flux given by eqn. (7.27), and obtain thereby an integral expression for the next-order iteration for the radiative flux, \(H^{(1)}(\tau)\).

b. Evaluate the required integral using properties of the exponential integrals, and show thereby that \(H^{(1)}(\tau)\) is not constant.

c. Using your favorite analysis and plotting software, e.g. Maple or Mathematica, plot the relative flux error \(H^{(1)}/H - 1\) vs. \(\tau\) for \(\tau = 0\) to \(\tau = 5\).

d. What is the maximum error, and at about what optical depth does it occur?

In principal one can continue to reapply the Lambda operator to get a sequence of higher iterations of \(J\), but the process requires difficult integrals involving product of exponential integrals, and moreover converges very slowly. To see this, let us focus on the Hopf function \(q(\tau)\), and specifically assume that some guess for this differs from the “exact” solution by a
constant, i.e. \( q(t) = q_{\text{exact}}(t) + C \). Application of the Lambda operator to both sides gives

\[
\Lambda_\tau[t + q(t)] = \Lambda_\tau[t + q_{\text{exact}}(t) + C] \tag{8.19}
\]
\[
\tau + q^{(1)}(\tau) = \tau + q_{\text{exact}}(\tau) + \Lambda_\tau[C] \tag{8.20}
\]
\[
\Delta q^{(1)}(\tau) \equiv q^{(1)} - q_{\text{exact}} = C(1 - E_2(\tau)/2) \tag{8.21}
\]

From this we see that the new error is reduced by half at the surface, i.e. \( \Delta q^{(1)}(0) = C/2 \), whereas at large optical depth there is essentially no improvement, i.e. \( \Delta q^{(1)}(\infty) = C \). Physically, this failure to improve the solution much beyond the surface can be traced to the fact that a photon mean-free-path corresponds to \( \Delta \tau = 1 \), which then essentially represents the depth of influence for each \( \Lambda \)-iteration. As such the full convergence of \( \Lambda \)-iteration is very slow, especially at great depth.

### 8.5. Isotropic, Coherent Scattering + Thermal Emission/Absorption

In this context of a gray atmosphere, a formally similar, but conceptually distinct, form for radiation transport arises in the case when the radiation remains “self-contained” within a single “coherent” frequency \( \nu \). The opacity in this case need not be gray, and indeed, we will see below such an approach can provide a first approximation for treating spectral lines with a highly frequency-dependent opacity. Nonetheless, such coherent frequency transport has many similarities to the gray case developed above.

As a specific example, consider a case in which the source function is given by

\[
S_\nu(\tau_\nu) = \epsilon B_\nu(\tau_\nu) + (1 - \epsilon) J_\nu(\tau_\nu). \tag{8.22}
\]

The first term on the right represents thermal emission for a (now assumed known) depth variation of the Planck function \( B_\nu(\tau_\nu) \). The second terms represents isotropic, coherent\(^8\) scattering that depends on the mean intensity \( J_\nu \) at the same frequency \( \nu \), for which the optical depth variation must be derived from a global solution of the scattered radiation transport.

Let us thus write the flux and K-moments of the tranfer equation as

\[
\frac{dH_\nu}{d\tau_\nu} = J_\nu - S_\nu = \epsilon(J_\nu - B_\nu) \tag{8.23}
\]

\(^8\)“Coherent” here refers to frequency, i.e. that the energy and thus frequency of the photon is not changed through the scattering process. In practice, line-scattering by ions leads to a “frequency redistribution” associated with the Doppler effect from thermal motions of the ions. We will return to this in our discussion of line-scattering below.
and

\[ \frac{dK_\nu}{d\tau_\nu} = H_\nu = \frac{1}{3} \frac{dJ_\nu}{d\tau_\nu}, \quad (8.24) \]

where the last equality again assumes the Eddington approximation \( J_\nu = 3K_\nu \). If we next assume a Planck function that is linear in optical depth, i.e.

\[ B_\nu(\tau_\nu) = a + b\tau_\nu, \quad (8.25) \]

we can combine these first-order ODE’s into a single second-order one,

\[ \frac{1}{3} \frac{d^2}{d\tau_\nu^2} (J_\nu - B_\nu) = \epsilon (J_\nu - B_\nu). \quad (8.26) \]

This can be readily integrated to give

\[ J_\nu - B_\nu = \alpha e^{-\sqrt{3}\epsilon\tau_\nu} + \beta e^{\sqrt{3}\epsilon\tau_\nu}, \quad (8.27) \]

where \( \alpha \) and \( \beta \) are integration constants. Since we know the solution is bounded at large optical depth, we must have \( \beta = 0 \). The other boundary condition comes from the gray-atmosphere surface condition \( J_\nu(0) = \sqrt{3}H_\nu(0) = (1/\sqrt{3})(dJ_\nu/d\tau_\nu)_0 \), which here implies \( a + \alpha = b/\sqrt{3} - \alpha\sqrt{\epsilon} \), or

\[ \alpha = \frac{b/\sqrt{3} - a}{1 + \sqrt{\epsilon}}. \quad (8.28) \]

The fully analytic solution for the mean intensity is thus

\[ J_\nu(\tau_\nu) = a + b\tau_\nu + \frac{b/\sqrt{3} - a}{1 + \sqrt{\epsilon}} e^{-\sqrt{3}\epsilon\tau_\nu}. \quad (8.29) \]

Eqn. (8.29) quantifies nicely the above physical arguments about thermalization in an atmosphere with non-zero scattering. For example, note that the LTE condition \( J \approx B \) is generally only recovered for optical depths of order the thermalization depth, i.e. for \( \tau_\nu \gtrsim \tau_{th} \approx 1/\sqrt{\epsilon} \).

In contrast, at the \( \tau_\nu = 0 \) surface we find, for the simple case of an isothermal atmosphere with \( b = 0 \) and so \( B_\nu = a = \text{constant} \), that the mean intensity is

\[ J_\nu(0) = \frac{\sqrt{\epsilon}}{1 + \sqrt{\epsilon}} B_\nu. \quad (8.30) \]

For \( \epsilon \ll 1 \), with thus a substantial level of scattering, we obtain \( J_\nu/B_\nu \ll 1 \), reflecting the excessive “leakage” of radiative energy due to the diffusive loss to empty space.
Figure 8.1.— $J_\nu - B_\nu$ vs. $\log \tau_\nu$ for analytic mean-intensity solution (8.29), assuming $a = 2$, $b = 3$, and various absorption fractions $\epsilon$.

Exercise: Assume a two-stream approximation in which the radiation field is characterized by the specific intensities $I_\nu^\pm$ along just two discrete directions $\mu = \pm \mu_1$, where $\mu_1$ is fixed.

a. Write down the discrete angle forms for both the mean intensity $J_\nu$ and the flux $H_\nu$ in terms of the $I_\nu^\pm$ and $\mu_1$.

b. By adding and subtracting the separate transfer equations for $I_\nu^+$ and $I_\nu^-$, derive equations for $dJ_\nu/d\tau_\nu$ and $dH_\nu/d\tau_\nu$.

c. Now combine these to get a second-order equation for $J_\nu$. For what value of $\mu_1$ does this become equivalent to the second-order equation for $J_\nu$ in the Eddington approximation?
d. For a finite slab of optical thickness \( \tau_s = 2\tau_m \), write out boundary conditions for the \( I^{\pm} \) at the appropriate surfaces \( \tau_\nu = 0 \) and \( \tau_\nu = \tau_s \).

e. Assume a source function of the form \( S_\nu = \epsilon B_\nu + (1 - \epsilon) J_\nu \), appropriate to coherent scattering plus thermal emission. For the case \( B_\nu = \text{constant} \) and \( \epsilon = \text{constant} \), use symmetry arguments to replace one of the surface B.C.’s with a B.C. at the slab midpoint \( \tau = \tau_m \). Then derive an expression for the ratio \( J/B \) at the slab midpoint in terms of \( \epsilon \) and \( \tau_m \).

f. A slab for which this midpoint ratio is approximately unity is called \textit{effectively thick}. What then is the requirement for such a slab to be “effectively thick”? Compare this to the requirement for the slab to be “optically thick”.

9. Line Opacity and Broadening

9.1. Einstein Relations for Bound-Bound Emission and Absorption

Let us now focus on cases wherein the opacity has a significant contribution from bound-bound processes. Because this leads to narrow “lines” of modified flux or intensity in a stellar spectrum, the overall process is call “line transfer”. There are 3 basic kinds of line processes associated with bound-bound transitions of atoms or ions:

1. \textit{Direct Absorption.} In which the absorbed photon induces a bound electron to go into another bound level at higher energy.

2. \textit{Spontaneous Emission.} In which an electron in a higher energy level spontaneously decays to lower level, emitting the energy difference as a photon.

3. \textit{Stimulated Emission.} In which an incoming photon induces an electron in a higher energy level to decay to a lower level, emitting in effect a second photon that is nearly identical in energy (and even phase) to the original photon.

For lower and upper levels \( i \) and \( j \), we can associate with these three processes the Einstein coefficients, written \( B_{ij} \), \( A_{ji} \), and \( B_{ji} \). The first of these is related to the opacity \( \kappa_\nu \) via the associated energy removed from an intensity beam \( I_\nu \),

\[
\rho \kappa_\nu I_\nu \equiv n_i B_{ij} \frac{h\nu_{ij}}{4\pi} \phi_\nu I_\nu ,
\]  

(9.1)

where \( n_i \) is the number of atoms or ions in the lower atomic state \( i \), and \( \phi_\nu \) is called the \textit{profile function}, defining just how the opacity varies for frequencies \( \nu \) near the resonance.
(a.k.a. “line-center”) value $\nu_{ij}$. It is normalized to unity when integrated over all frequencies,
\[
\int_0^\infty \phi_\nu \, d\nu = 1. \tag{9.2}
\]

In the idealized case that both the upper and lower energy levels are infinitesmally sharp and well defined, the profile function measured in the rest-frame of the atom can be written as a Dirac delta function, $\phi_\nu = \delta(\nu - \nu_{ij})$. In practice, the Heisenberg uncertainty principal means a level with a finite lifetime has energy uncertainty, giving the profile what is known as “natural broadening”. Moreover, the perturbative effect of other atoms and ions leads to a kind of “pressure” (a.k.a. “Stark”) broadening. These both act on the intrinsic profile in the atom’s frame, but if one accounts for the random thermal motion of atoms, then the Doppler effect leads to an additional “thermal Doppler broadening” for the profile measured in the rest frame of the overall stellar atmosphere. We discuss these further below.

The Einstein coefficients for emission have to be divided between spontaneous and stimulated components,
\[
\eta_\nu^{\text{sp}} \equiv n_j A_{ji} \frac{h \nu_{ij}}{4\pi} \psi_\nu, \tag{9.3}
\]
and
\[
\eta_\nu^{\text{stim}} \equiv n_j B_{ji} \frac{h \nu_{ij}}{4\pi} \psi_\nu I_\nu, \tag{9.4}
\]
where $n_j$ is the number density of ions in the upper level $j$, and $\psi_\nu$ is the emission profile. In practice, one can often assume $\psi_\nu = \phi_\nu$, which implies a randomization or complete redistribution (CRD)$^9$ of the photon frequencies between absorption and emission within the line profile $\phi_\nu$.

To proceed, let us consider the case of a gas in strict thermodynamic equilibrium (TE), with $I_\nu = B_\nu$, and the ratio of the population in upper and lower levels set by the Boltzmann equilibrium condition,
\[
\frac{n_j^*}{n_i^*} = \frac{g_j}{g_i} e^{-h \nu_{ij}/kT}, \tag{9.5}
\]
where $g_i$ and $g_j$ are the statistical weights of the lower and upper levels, and the asterisks emphasize that we’re specifically referring to TE level populations. The exponential term is

$^9$More generally, one needs to consider a frequency redistribution function $R(\nu, \nu') \equiv \phi_\nu \psi_{\nu'}$, describing the probability that absorption of a photon of initial frequency $\nu$ is followed by reemission of a photon of different frequency $\nu'$. In CRD, $R(\nu, \nu') = \phi_\nu \phi_{\nu'}$. Our previous assumption of coherent scattering, for which the frequency of emitted photon exactly equals that of the absorbed photon, amounts to taking $R(\nu, \nu') = \phi_\nu \delta(\nu' - \nu)$. 
the “Boltzmann factor”, with eqn. (9.5) then known as the “Boltzmann relation”. It is one of the fundamental underpinnings of thermodynamics and statistical mechanics.

In TE, the principal of detailed balance requires \( \phi_\nu = \psi_\nu \). It also requires that the energy absorbed be equal to that emitted,

\[
n_i^* B_{ij} B_\nu = n_j^* (A_{ji} + B_{ji} B_\nu) ,
\]

Solving for the Planck function, we find

\[
B_\nu = \frac{A_{ji} / B_{ji}}{g_i B_{ij} e^{h\nu/kT} - 1} = \frac{2h\nu^3 / c^2}{e^{h\nu/kT} - 1} .
\] (9.7)

But because the Einstein coefficients don’t depend on temperature or, for that matter the conditions of LTE, in order to recover the proper form for the Planck function, we must require these two Einstein relations:

\[
A_{ji} = \frac{2h\nu_{ij}^3}{c^2} B_{ji} ,
\] (9.8)

and

\[
g_i B_{ij} = g_j B_{ji} .
\] (9.9)

It should be emphasized again that these Einstein relations are quite general, and do not depend on an assumption of LTE. At a base level, they really stem from the time-reversal nature of microscopic laws of physics, since running a clock backwards on an absorption looks like an emission, and vice versa. It is the basis of the general rule of thumb: “a good absorber is a good emitter”. In general then, once we know the absorption coefficient \( B_{ij} \), we can use these Einstein relations to obtain the stimulated and spontaneous emission coefficients \( B_{ji} \) and \( A_{ji} \).

### 9.2. The Classical Oscillator

A key issue in line-transfer is computing the line-absorption opacity,

\[
\kappa_\nu = \frac{n_i}{\rho} \sigma_\nu ,
\] (9.10)

in terms of the level population \( n_i \) for the lower level \( i \), and the associated bound-bound cross section,

\[
\sigma_\nu = \sigma_{tot} \phi_\nu ,
\] (9.11)
where $\sigma_{\text{tot}}$ is the total, \textit{frequency-integrated} cross section, with units cm$^2$ Hz. For complex atoms with many electrons and energy levels, it can be quite difficult to calculate these cross sections, and often they are best determined by experiment.

But within the context of a basic classical model of an atom, one can derive a quite simple scaling, known as the \textit{“classical oscillator”}, for which the frequency-integrated cross section is just

$$
\sigma_{\text{cl}} \equiv \frac{\pi e^2}{m_e c} = \pi r_e c,
$$

(9.12)

where again $e$ and $m_e$ are the electron charge and mass, and the latter equality casts this in terms of the classical electron radius $r_e = e^2/m_e c^2$. More complete calculations based on modern quantum mechanics are generally written with the integrated cross sections scaled by this classical oscillator,

$$
\sigma_{\text{tot}} = f_{ij} \sigma_{\text{cl}} = B_{ij} \frac{\hbar \nu_{ij}}{4\pi},
$$

(9.13)

where $f_{ij}$ is a dimensionless \textit{“oscillator strength”}, typically of order unity for quantum mechanically allowed transitions, but very small for \textit{“forbidden”} transitions that violate some first-order selection rule. The latter equality above shows the relationship to the Einstein absorption coefficient $B_{ij}$.

Collecting these relations together, we have for the frequency-dependent line-opacity,

$$
\kappa_\nu = \frac{n_i}{\rho} f_{ij} \sigma_{\text{cl}} \phi_\nu.
$$

(9.14)

\subsection*{9.3. Gaussian Line-Profile for Thermal Doppler Broadening}

Let us now derive a form for the line-profile function $\phi_\nu$ that results from the Doppler broadening by the thermal motion of atoms in an (otherwise static) atmosphere. If $v$ is the speed of an atom’s thermal motion in the direction of a photon with rest frequency $\nu$, then by the standard formula for Doppler shift, the frequency in the atom’s frame is

$$
\nu_a = \nu (1 - v/c).
$$

(9.15)

For gas of temperature $T$, the mean kinetic energy due to random thermal motion is $kT = \frac{mv_{th}^2}{2}$, where $v_{th}$ is the average thermal speed, and $m$ is the mass of the absorbing atom. The fraction of atoms in a speed interval between $v$ and $v + dv$ is then given by a simple
Gaussian distribution,\(^{10}\)

\[
f(v)dv = \frac{e^{-(v/v_{th})^2}}{\sqrt{\pi v_{th}}} dv,
\]
(9.16)

The line-profile in the star’s frame \(\phi_\nu\) can be obtained by convolving this distribution with the profile in the atom’s frame \(\phi_{\nu_a}\).

For the idealized case that the atomic-frame profile is represented by a Dirac delta-function \(\delta(\nu_a - \nu_o)\), the convolution integral becomes quite straightforward to evaluate,

\[
\phi_D(\nu) = \int_{-\infty}^{\infty} \delta[\nu(1 - v/c) - \nu_o] \frac{e^{-(v/v_{th})^2}}{\sqrt{\pi v_{th}}} dv,
\]
(9.17)

giving

\[
\phi_D(\nu) = \frac{e^{-(\nu-\nu_o)^2/\Delta\nu_D^2}}{\sqrt{\pi \Delta\nu_D}},
\]
(9.18)

where

\[
\Delta\nu_D = \frac{v_{th} \nu_o}{c} = \sqrt{\frac{2kT}{m}} \frac{\nu_o}{c}
\]
(9.19)

represents a characteristic thermal Doppler width for the line. Thus lines from a gas with finite temperature are broadened by \(\pm \Delta\nu_D\) on both the lower (red) and higher (blue) frequency side of line-center frequency \(\nu_o\).

**Exercise:** Fill in the steps in the integral evaluation between (9.17) and (9.18), by making variable substitutions and accounting explicitly for the dimensions of the delta function. Then integrate eqn. (9.18) over all frequencies to confirm that the Doppler profile \(\phi_D(\nu)\) has the proper unit normalization from eqn. (9.2).

Sometimes the profile function is instead defined in terms of the photon wavelength \(\lambda\) instead of frequency \(\nu\). But to keep the proper units and normalization, note that one requires \(\phi_\nu d\nu = \phi_\lambda d\lambda\). It is often convenient to write a thermally broadened line-profile in terms of thermal Doppler widths \(x\) from line center,

\[
\phi_D(x) \equiv \frac{e^{-x^2}}{\sqrt{\pi}}.
\]
(9.20)

---

\(^{10}\)This is another application of the Boltzmann distribution discussed above, which says that the distribution of states of energy \(E\) is proportional to \(e^{-E/kT}\), where here the energy is just the kinetic energy of the individual atoms of speed \(v\), i.e. \(E = mv^2/2\), with the thermal speed given by \(v_{th} = \sqrt{2kT/m}\).
Depending on the context, the variable $x$ can either be in wavelength, $x_\lambda = (\lambda/\lambda_o - 1)c/v_\text{th} = (\lambda - \lambda_o)/\Delta\lambda_D$ or in frequency, $x_\nu = (\nu - \nu_o)/\Delta\nu_D$. But for the usual case that $\Delta\lambda_D/\lambda_o = \Delta\nu_D/\nu_o = v_\text{th}/c \ll 1$, these two definitions are just related by a simple sign flip.

**Exercise:** For narrow lines characterized by fractional width $v_\text{th}/c \ll 1$, show that indeed $x_\lambda \approx -x_\nu$.

For hydrogen atoms at the solar effective temperature $T = 5800$ K, $v_\text{th} \approx 7$ km/s, and even in hotter stars the dominant lines from partially ionized “metals” typically have $v_\text{th} \sim 10$ km/s. Comparing this to the speed of light $c = 3 \times 10^5$ km/s, we conclude that the fractional thermal width of lines is thus typically of order

$$\frac{\Delta\lambda_D}{\lambda_o} = \frac{\Delta\nu_D}{\nu_o} = \frac{v_\text{th}}{c} \approx 3 \times 10^{-5}. \quad (9.21)$$

### 9.4. The Resonant Nature of Bound vs. Free Electron Cross Sections

The concentration of line-opacity into such very narrow segments in frequency or wavelength is one key factor is making line absorption strong relative to continuum processes at frequencies near an atomic resonance. But this is not the whole, or even main, reason for the inherent strength of line opacity. In particular, even if one averages the total line cross section over a much larger frequency interval given by its own resonance frequency, i.e. $\sigma_{\text{tot}}/\nu_o$, this still turns out to be much greater than a characteristic continuum cross section, like the Thomson cross section $\sigma_{\text{Th}}$. For example, for an allowed line transition with oscillator strength $f_{ij} \approx 1$, we can define a characteristic line-strength ratio $Q_\lambda$ in terms the classical oscillator $\sigma_{cl}$,

$$Q_\lambda \equiv \frac{\sigma_d}{\nu_o \sigma_{\text{Th}}} = \frac{\pi r_e c}{\nu_o (8/3)\pi r_e^2} \quad (9.22)$$

$$= \frac{3 \lambda_o}{8 r_e} \quad (9.23)$$

$$= 7.5 \times 10^8 \lambda_{5000}, \quad (9.24)$$

where $r_e = e^2/m_ec^2 \approx 2.5 \times 10^{-13}$ cm is the classical electron radius discussed previously. The last equality shows that, in terms of a wavelength scaled by a typical optical value, i.e. $\lambda_{5000} = \lambda/5000$ Å, this ratio is very large. By this measure, one can thus think of line cross sections as being roughly a billion times stronger than for electron scattering!

The basic physical reason for this great intrinsic strength of lines lies in the ability of bound electrons to resonate with narrow bands of the incident radiation, greatly increasing
the total cross section. This is much the same principle that makes a whistle loud, with a response tuned to a specific frequency, in contrast to the softer, broadband noise from just blowing into open air. Indeed, by next examining the finite lifetime and damping of the resonance, we will see that the ratio \( Q_\lambda \) is closely related to quality of the resonance\(^{11}\).

9.5. Frequency Dependence of Classical Oscillator: the Lorentz Profile

This simple classical model of a bound oscillator can also be used to derive the frequency-dependent cross section, measured in the rest frame of the atom, \( \sigma_\nu \). Multiplied by the oscillator strength \( f_{ij} \) to account for quantum mechanical effects, the result is (from Mihalas 1968, eq. [4-32]):

\[
\sigma_\nu = f_{ij} \sigma_{Th} \frac{\nu^4}{(\nu^2 - \nu_o^2)^2 + \nu^2(\Gamma / 2\pi)^2},
\]

(9.25)

where

\[
\Gamma = \frac{8\pi^2}{3} \frac{r_e \nu_o^2}{c} = \frac{g_j}{g_i} \frac{A_{ji}}{3f_{ij}}
\]

(9.26)

is a rate parameter for the damping of the oscillator. The latter equality makes use of the Einstein relations and eqn. (9.13) to show this classical damping rate is closely related to the quantum mechanical transition rate \( A_{ji} \).

Eqn. (9.25) yields physically different behavior over three regimes in \( \nu \):

1. Thomson scattering \((\nu \gg \nu_o)\). For high frequencies well away from the resonance, we recover (with \( f = 1 \)) the simple Thomson cross section for free electron scattering,

\[
\sigma_\nu \approx \sigma_{Th}; \quad \nu \gg \nu_o.
\]

(9.27)

2. Rayleigh scattering \((\nu \ll \nu_o)\). In the opposite limit of low frequency well below the resonance, we obtain the Rayleigh scattering limit, with

\[
\sigma_\nu \approx \sigma_{Th} \left( \frac{\nu}{\nu_o} \right)^4; \quad \nu \ll \nu_o.
\]

(9.28)

\(^{11}\) I am indebted to Ken Gayley for first pointing out to me this very deep insight into the resonant nature of line absorption. The discussion in this and the next two subsections is largely taken from unpublished notes he shared with me. But for some interesting applications of this general property in the context of radiative driving in stellar winds, I highly recommend Gayley (1995, ApJ 454, 410).
The strong frequency dependence of Rayleigh scattering leads to a substantially red- 
denng of a light source, with the scattered, diffuse light dominated by bluer color. In the context of scattering by molecules in the earth’s atmosphere, such Rayleigh scattering makes the sky blue, and leads to the distinct redness of the solar disk at sunset.

3. **Line absorption** ($\nu \approx \nu_o$). When $\nu$ is very close to $\nu_o$, we obtain a line-opacity of the form

$$\sigma_\nu = f_{ij} \sigma_{cl} \phi_L(\nu),$$

(9.29)

where

$$\phi_L(\nu) = \frac{\Gamma / 4\pi^2}{(\nu - \nu_o)^2 + (\Gamma / 4\pi)^2}$$

(9.30)

is known as the normalized Lorentz profile.

**Exercise:** Derive the Lorentz profile (9.30) from the general eqn. (9.25) in the case that $\nu \approx \nu_o$. *Hint:* Note that in this case, $(\nu^2 - \nu_o^2)^2 \approx 4\nu_o^2(\nu - \nu_o)^2$.

Note that *all three* regimes are present for any resonator, and free electrons can be treated simply by taking $\nu_o$ and $\Gamma$ to be zero, with unit oscillator strength, in eqn. (9.25). This suggests that eqn. (9.25) can be rewritten in a simple approximate form that underscores these points,

$$\sigma_\nu \approx f_{ij} \sigma_{Th} \min\{1, \nu^4 / \nu_o^4\} + f_{ij} \sigma_{cl} \phi_L(\nu).$$

(9.31)

Eqn. (9.31) allows us to examine the relative importance of bound and free electrons. It shows that the Thomson cross section is present even for bound electrons. Thus the use of the Lorentz profile is actually an approximation to eqn. (9.25) in the vicinity of the resonance. Using eqn. (9.25) instead eliminates the confusion between the units of $\sigma_{Th}$ and $\sigma_{cl}$, and it is readily seen from eqn. (9.31) that the presence of a resonance merely enhances the cross section in the vicinity of $\nu_o$, and sets up a wide Rayleigh regime where the cross section is somewhat reduced. These are constructive and destructive interference effects, respectively, and the constructive effects far outweigh the destructive ones for a flat continuum. The oscillator strength is an overall multiple that applies in all three regimes, and effectively gives the probability that the oscillator in question is quantum mechanically realized.

### 9.6. The High “Quality” of Line Resonance

As shown in eqn. (9.26), the damping rate $\Gamma$ is closely related to the transition rate $A_{ji}$, meaning $1/\Gamma$ effectively characterizes the lifetime of the state, or equivalently, the duration
of the resonance. As such, we can readily define the classical "Q" or quality of the resonance,

\[ Q = \frac{\nu_0}{\Gamma} = \frac{3}{8\pi^2} \frac{\lambda_0}{r_e} = \frac{Q\lambda}{\pi^2} \approx 8 \times 10^7 \lambda_{5000}, \]  

(9.32)

which effectively gives the number of cycles required to damp the oscillation after external driving is turned off. It also gives the number of cycles over which the oscillator can retain phase coherence, which measures its potential for constructive interference. This interference allows the bound electrons to dominate the free electrons by essentially the factor \( QA \), where \( A \) is the relative fraction of bound vs. free electrons in the stellar atmosphere.

For example, in the atmosphere of a cool star like the sun, most electrons are still bound to hydrogen, and so photons with a frequency near the resonance for one of the bound-bound transitions of a hydrogen, the opacity can be enormous compared to electron scattering. Hydrogen lines in the sun are thus typically very strong, even greater than \( Q \sim 10^8 \) times the opacity for free electrons.

On the other hand, in hotter stars, hydrogen is fully ionized, and the dominant lines come from various incomplete ionization states of much less abundant "metals", e.g. carbon, nitrogen, oxygen and iron. With relative abundance of bound electrons thus of order \( A \sim 10^{-4} \), the strengths of lines relative to electron scattering is somewhat reduced, but still large, \( QA \sim 10^3 \).

9.7. The Voigt Profile for Combined Doppler and Lorentz Broadening

The above shows that the intrinsic profile in the atom’s frame is not completely sharp like a delta-function, but rather, due to the finite lifetime of the state, has an inherent broadening characterized by the Lorentz profile \( \phi_L(\nu) \), as given by eqn. (9.30). To obtain the overall profile in the frame of the stellar atmosphere, we must convolve this Lorentz profile with the Gaussian distribution of speeds that give thermal Doppler broadening,

\[ \phi_V(\nu) = \int_{-\infty}^{\infty} \phi_L[\nu(1-v/c)] \frac{e^{-(v/v_{th})^2}}{\sqrt{\pi}v_{th}} \, dv. \]  

(9.33)

For such a “Voigt profile”, it is traditional to define a parameterized Voigt function,

\[ H(a_v, x) = \frac{a_v}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(x-y)^2 + a_v^2} \, dy \]  

(9.34)

\[ = \Re \left[ e^{a_v+ix} \text{erfc}(a_v + ix) + e^{a_v-ix} \text{erfc}(a_v - ix) \right], \]  

(9.35)
Fig. 9.1.— \( \log[\phi(x)] \) vs. \( x \) for Voigt (heavy solid), Doppler (medium dashed) and Lorentz (light dotted) profiles with various damping parameters \( a_v \). The intersection points \( x_* \) at which \( \phi_D(x_*) \equiv \phi_L(a_v, x_*) \) mark the transition frequency between Doppler core and Lorentz damping wings. The light solid curves show that the sum of the Doppler and Lorentz profiles gives a good, simple approximation to the full Voigt profile for \( x > 1 \), as noted in eqn. (9.41).

where the latter analytic form was obtained from Mathematica. Here we have again used a Doppler-unit frequency difference from line-center,

\[
x \equiv \frac{\nu - \nu_0}{\Delta \nu_D},
\]

and we have now defined a scaled damping parameter,

\[
a_v \equiv \frac{\Gamma}{4\pi \Delta \nu_D},
\]
which is typically quite small, $a_v \sim 10^{-4} \ll 1$. In these scaled units, the Doppler profile is given by eqn. (9.20), while the Lorentz profile is

$$\phi_L(a_v, x) = \frac{a_v/\pi}{x^2 + a_v^2}. \quad (9.38)$$

Figure 9.1 compares log plots of the frequency variation of Voigt, Doppler, and Lorentz profiles, for various values of $a_v$.

**Exercise:** Show that the dimensionless damping parameter $a_v$ can be written in terms of the resonance quality $Q$ defined in eqn. (9.32), and the ratio of thermal speed to light speed $v_{th}/c$. For the typical values quoted above for $Q$ of an allowed transition with wavelength in the optical, and for thermal speed $v_{th}$ in a stellar atmosphere, estimate an associated numerical value of $a_v$, confirming thereby that it is indeed very small, $a_v \ll 1$.

We can now write a Doppler-unit Voigt profile

$$\phi_v(a_v, x) = \frac{H(a_v, x)}{\sqrt{\pi}} \quad (9.39)$$

$$\approx \phi_D(x) \quad ; \quad |x| \leq 1 \quad (9.40)$$

$$\approx \phi_D(x) + \phi_L(a_v, x) \quad ; \quad |x| > 1. \quad (9.41)$$

Figure 9.1 compares the full Voigt profile with Doppler and Lorentz profiles, and also their sum. As illustrated by the vertical dashed lines, the frequency $x_*$ — defined implicitly as the outer root of $\phi_D(x_*) = \phi_L(a_v, x_*)$ — marks the transition from the Doppler core to the Lorentz damping wings. This can be solved explicitly using the “Lambert” or “ProductLog” function\(^{12}\), but for the usual case of a very small damping parameter, $a_v \ll 1$, figure 9.1 shows that typically $x_* = 2 - 4$. Using $x_* \approx 3$ as first approximation, we can write an explicit next approximation in the form

$$x_* \approx \sqrt{\ln(3\sqrt{\pi}/a_v)}. \quad (9.42)$$

For example, for a typical value $a_v = 10^{-4}$, this approximation gives $x_* \approx 3.3$, whereas the exact solution is $x_* = 3.5$.

---

10. Classical Line Transfer: Milne-Eddington Model for Line + Continuum

Let us now derive the emergent intensity and/or flux from an atmosphere in which the opacity has contributions from both line and continuum. For this, note that in §8.5 we have already solved the radiative transfer for thermal emission plus coherent, scattering at a single, fixed, isolated frequency \( \nu \). In that solution, no restriction was made for what the source of the opacity might be at the chosen frequency, and so we are free now to specify this, using both the continuum and line opacity sources we have been examining above.

So consider now a case where at the given frequency \( \nu \) the total opacity \( k_\nu \) stems from a combination of both continuum and line processes,

\[
k_\nu = k_c + k_l \phi_\nu ,
\]

(10.1)

with also corresponding emissivities

\[
\eta_\nu = \eta_c + \eta_l \phi_\nu .
\]

(10.2)

Let us further assume that these opacities are divided between scattering and absorption components, with total opacities and associated photon destruction probabilities given by

\[
k_c = \kappa_{cs} + \kappa_{ca}
\]

(10.3)

\[
k_l = \kappa_{ls} + \kappa_{la}
\]

(10.4)

\[
\epsilon_c = \kappa_{ca} / k_c.
\]

(10.5)

\[
\epsilon_l = \kappa_{la} / k_l.
\]

(10.6)

We can then define a frequency-dependent destruction probability

\[
\epsilon_\nu = \frac{\epsilon_c + \epsilon_l \beta_\nu}{1 + \beta_\nu} ,
\]

(10.7)

where the relative strength of the total line to continuum opacity is defined by

\[
\beta_\nu = \frac{k_l}{k_c} \phi_\nu .
\]

(10.8)

This still gives a source function of the form in eqn. (8.22),

\[
S_\nu(\tau_\nu) = \epsilon_\nu B_\nu(\tau_\nu) + (1 - \epsilon_\nu) J_\nu(\tau_\nu) .
\]

(10.9)

As in §8.5, let us assume a Planck function that is linear in optical depth, defined now by the continuum opacity,

\[
B_\nu(\tau_c) = a + b \tau_c
\]

\[
\equiv a + p_\nu \tau_\nu ,
\]

(10.10)

(10.11)
where the second equality gives the variation with frequency-dependent optical depth through the coefficient
\[ p_{\nu} \equiv b \frac{k_c}{k_{\nu}} = \frac{b}{1 + \beta_{\nu}}. \] (10.12)

An essential point here is that, with just these redefinitions, the solution is entirely analogous to that given by eqn. (8.29),
\[ J_{\nu}(\tau_{\nu}) = a + p_{\nu} \tau_{\nu} + \frac{p_{\nu}/\sqrt{3} - a}{1 + \sqrt{\epsilon_{\nu}}} e^{-\sqrt{3} \epsilon_{\nu} \tau_{\nu}}. \] (10.13)

We can use this to obtain the emergent flux
\[ H_{\nu}(0) = J_{\nu}(0) \sqrt{3} = \frac{1}{3} \frac{\sqrt{3} \epsilon_{\nu} a + p_{\nu}}{1 + \sqrt{\epsilon_{\nu}}}. \] (10.14)

For continuum opacity, we have $\beta_{\nu} \to 0$, $\epsilon_{\nu} = \epsilon_c$, and $p_{\nu} = b$, yielding
\[ H_c(0) = \frac{1}{3} \frac{\sqrt{3} \epsilon_c a + b}{1 + \sqrt{\epsilon_c}}. \] (10.15)

The ratio of these gives the residual flux of the line,
\[ R_{\nu} \equiv \frac{H_{\nu}(0)}{H_c(0)} = \left( \frac{\sqrt{3} \epsilon_{\nu} a + b/(1 + \beta_{\nu})}{\sqrt{3} \epsilon_c a + b} \right) \left( \frac{1 + \sqrt{\epsilon_c}}{1 + \sqrt{\epsilon_{\nu}}} \right). \] (10.16)

This general result contains interesting behaviors in various special cases, as we now explore.

10.1. Scattering line with thermal continuum

Let us first consider the case of a pure-scattering line, with $\epsilon_l = 0$, coupled with a pure-absorption, thermal continuum, $\epsilon_c = 1$, which together also imply $\epsilon_{\nu} = 1/(1 + \beta_{\nu})$. We then find
\[ R_{\nu} = 2 \frac{\sqrt{3} a + b/\sqrt{1 + \beta_{\nu}}}{(\sqrt{3} a + b) (1 + \sqrt{1 + \beta_{\nu}})}. \] (10.17)

Note then that for a very strong scattering line, i.e. with $\beta_{\nu} \to \infty$, we obtain $R_{\nu} \to 0$, meaning it becomes completely saturated or dark.
10.2. Absorption line with thermal continuum

For the case of pure-absorption in both continuum and line, with \( \epsilon_c = \epsilon_l = 1 \), eqn. (10.16) becomes

\[
R_\nu = \frac{\sqrt{3}a + b/(1 + \beta_\nu)}{\sqrt{3} a + b}.
\]  

(10.18)

Now for the limit of strong absorption line, \( \beta_\nu \to \infty \), we find

\[
R_\nu = R_o = \frac{1}{1 + b/\sqrt{3}a},
\]  

(10.19)

which is generally nonzero. Very strong absorption lines thus do not become completely dark, even at line center.

![Graph](attachment:image.png)

Fig. 10.1.— The residual flux \( R_x \) vs. Doppler-scaled frequency \( x \) for the case of pure-absorption continuum \( \epsilon_c = 1 \) with \( b/a = 3/2 \), and Voigt line damping parameter \( a_v = 10^{-3} \). The left panel is for pure-absorption lines (\( \epsilon_l = 1 \)) and the right is for pure-scattering lines (\( \epsilon_l = 0 \)), with the overplots showing profiles for central line strengths from \( \beta_\nu = 1 \) to \( \beta_o = 10^6 \) in steps of factor 10. Note that the strong scattering lines become black at line center, while absorption lines saturate to a level that depends on \( b/a \).

Figure 10.1 compares line profiles for this absorption case (left) with the scattering case above (right), for a Voigt parameter \( a_v = 10^{-3} \), and Eddington gray atmosphere values for the ratio \( b/a = 3/2 \). The overplots are for line-center strength \( \beta_o = 1 \) to \( 10^6 \) in steps of factor 10. Note that the strong scattering lines become black at line center, while absorption lines saturate to a level that depends on \( b/a \).
Strong lines have a dual profile character, with a saturated Doppler core, and broad, gradual Lorentz damping wings. Once lines saturate in core, the additional growth of absorption with increasing line opacity $\beta_o$ occurs through a very gradual (logarithmic) expansion of the core. But with further increase in $\beta_o$, there develops a stronger growth in absorption with the expansion of the damping wings. Further details are given in the “curve of growth” discussion below.

**Exercise:** Consider the analytic Milne-Eddington model for the case of a pure-absorption continuum ($\epsilon_c = 1$). Assume $\beta_\nu = \beta_o \phi_\nu(x)$ and $B_\nu(\tau_c) = a + b\tau_c$ with $b/a = 3/2$.

a. For pure-scattering lines ($\epsilon_l = 0$), plot the residual flux profiles $R_x$ vs. Doppler-unit frequency displacement $x$ over the range $[−10, 10]$, overplotting curves for $\beta_o = 100$ and $10^4$, and for $a_\nu = 10^{-4}$, $10^{-3}$, and $10^{-2}$.

b. Do the same for pure-absorption lines ($\epsilon_l = 1$).

c. For the case $\beta_o = 100$ and $a_\nu = 10^{-2}$, compute the source function $S_\nu$ at optical depth $\tau_\nu = 2/3$ for both a pure-scattering and a pure-absorption line. Use the Eddington-Barbier relation to relate these to the residual fluxes obtained for this case in parts (a) and (b). Do the same for the $\beta_o = 10^4$ case.

### 10.3. Absorption lines in a Gray Atmosphere

For a gray atmosphere, we have $S_c = J_c = B_\nu$, with

$$B_\nu(\tau) = B_\nu(T_o) + \frac{dB_\nu}{d\tau_c}\tau_c = a + b\tau_c,$$  \hspace{1cm} (10.20)

and [cf. eqn. (8.12)]

$$T^4 = T_{o}^4 \left(1 + \frac{3}{2}\tau_c\right),$$  \hspace{1cm} (10.21)

where $T_o \approx T_{\text{eff}} / 2^{1/4}$ is the surface temperature. From these relations we see that

$$\frac{b}{a} = \frac{3}{8} X_o = \frac{3}{8} \frac{h\nu/kT_o}{1 - e^{-h\nu/kT_o}}.$$  \hspace{1cm} (10.22)

For a strong absorption line in such a gray atmosphere, we find residual flux

$$R_o = \frac{1}{1 + \sqrt{3X_o/8}}.$$  \hspace{1cm} (10.23)
For example, \( T_o = 4800 \) K, and for \( \lambda \approx 5000 \) Å, we get \( X_o \approx 6 \), which applied to eqn. (10.23) gives \( R_o = 1/(1 + 3\sqrt{3}/4) = 0.44 \). This is in good agreement with the central depth of the Hydrogen Balmer line, \( \text{H}\alpha \), which as a “subordinate line” (originating from an upper level), behaves like almost like a pure absorption.

In contrast, strong “resonances lines”, which start at the ground level, tend to behave like scattering lines, and when very strong, they do indeed become nearly black at line center.

### 10.4. Center to Limb Variation of Line Intensity

The formal solution gives for the emergent intensity,

\[
I_\nu(\mu, 0) = \int_0^\infty \left[ B_\nu + (1 - \epsilon_\nu)(J_\nu - B_\nu) \right] e^{-t/\mu} dt/\mu
\]

\[
= a + p_\nu\mu + \frac{(p_\nu - \sqrt{3}a)(1 - \epsilon_\nu)}{\sqrt{3}(1 + \sqrt{\epsilon_\nu})(1 + \sqrt{3\epsilon_\nu}\mu)}.
\]

For pure-absorption continuum with \( \beta_\nu = 0 \) and \( \epsilon_c = 1 \), we find again \( I_c = a + b\mu \), from which we can now define the residual intensity profile

\[
r_\nu(\mu) \equiv \frac{I_\nu(\mu, 0)}{I_c(\mu)}.
\]

For a pure-absorption line with \( \epsilon_l = \epsilon_c = \epsilon_\nu = 1 \), we find

\[
r_\nu(\mu) \equiv \frac{a + b\mu/(1 + \beta_\nu)}{a + b\mu} \to 1 \quad \text{as} \quad \mu \to 0,
\]

from which we see that the absorption lines disappear at the limb, regardless of the value of \( \beta_\nu \).

By contrast, for a scattering line with \( \epsilon_l = 0 \) and thus \( \epsilon_\nu = 1/(1 + \beta_\nu) \), we find that for \( \beta_o \to \infty \), \( r_\nu(\mu) \to 0 \) for all \( \mu \). Thus strong scattering lines are dark all across the solar disk.

### 10.5. Schuster Mechanism: Line Emission from Continuum Scattering Layer

In a star’s spectrum, the lines most commonly appear in absorption, meaning the intensity or flux in the line is lower than in the nearby continuum. But occasionally lines can also appear in emission, with the intensity or flux higher than in the continuum. One of a handful of processes for producing emission lines is known as the Schuster mechanism, which results when the continuum opacity is dominated by scattering, such as from free electrons.
Fig. 10.2.— Line profiles for Schuster model for an LTE line ($\epsilon_l = 1$) in a pure-scattering continuum ($\epsilon_c = 0$). The Voigt parameter is fixed at $a_\nu = 10^{-4}$, and the line-center strength is set to $\beta_o = 10$ (left) or $\beta_o = 100$ (right). The overplots show results for various values for the ratio of constants $a/b$, illustrating how a large $a/b$ causes the line to go into emission.

Supposing that the continuum is pure scattering, $\epsilon_c = 0$, then $\epsilon_\nu = \epsilon_l \beta_\nu / (1 + \beta_\nu)$. Plugging these into eqn. (10.16), the resulting residual flux profile becomes

$$R_\nu = \frac{1}{1 + \beta_\nu} + \frac{\sqrt{3} \epsilon_\nu a/b}{1 + \sqrt{\epsilon_\nu}}.$$  

(10.28)

If the line opacity is also pure scattering, with $\epsilon_l = 0$, then the residual flux is given by

$$R_\nu = \frac{1}{1 + \beta_\nu} < 1,$$  

(10.29)

which thus is always in absorption, becoming black at line center in the strong line limit $\beta_o \to \infty$.

But for an absorption line with $\epsilon_l = 1$, we find for strong lines $\beta_o \to \infty$ near line center,

$$R_o \to \frac{\sqrt{3} a}{2 b},$$  

(10.30)

which, for a weak temperature gradient with small $b/a$, can exceed unity, implying a net line emission instead of absorption.

Physically this can be understood from the fact that scattering makes the continuum source function low near the surface, $S_c(0) = J_c(0) \ll B(0)$, which by the Eddington-Barbier relation implies a weak continuum flux. By comparison, the absorption nature of the line
means its surface source function is at the much higher Planck level, \( S_l(0) \approx B(0) \). This indicates the line can potentially be brighter, but only if the decline from the negative temperature gradient term is not too steep.

Figure 10.2 plots sample line profiles for a moderate and strong line (\( \beta_o = 10 \), left; and \( \beta_o = 100 \), right), assuming various values for the ratio \( a/b \). Note that for \( a/b < 1/\sqrt{3} \), the lines remain always in absorption, while for \( a/b > \sqrt{3}/2 \) they are always in emission. For the intermediate case, the profiles have a mixed character, with emission wings and a central reversal toward the line-core, sometimes even going below the continuum at line center. The critical ratio \( a/b = 5/2\sqrt{3} \) gives a case where the center of a strong line reverses just down to the continuum, with the rest of the line in emission.

**Exercise**

In the Schuster-Schwarzschild model, line formation is assumed confined to a finite “reversing layer”, of optical thickness \( T_\nu \), and illuminated from below by an incident intensity \( I_o \). In the reversing layer, there is no continuum absorption or scattering, and \( B_\nu(\tau_\nu) = a + b\tau_\nu \). Assume a two-stream approximation, in which \( I_\nu(\mu) = I_\nu^\pm \), with only two discrete \( \mu = \pm \frac{1}{2} \).

a. For pure, coherent, line-scattering, show that \( H_\nu \equiv \frac{1}{4}(I_\nu^+ - I_\nu^-) = \text{constant} = \frac{1}{4}I_o/(1 + T_\nu) \), and that \( J_\nu(\tau_\nu) = 2H_\nu(1 + 2\tau_\nu) \), where \( 0 \leq \tau_\nu \leq T_\nu \). Why are these results independent of the Planck function parameters \( a \) and \( b \)?

b. Now derive \( H_\nu \) and \( J_\nu \) for a pure absorption line. Contrast the dependence here on \( a \) and \( b \) with that from part (a), and briefly discuss the physical reasons for the difference.

11. Curve of Growth of Equivalent Width

11.1. Equivalent Width

Because lines are so typically very narrow, with fractional widths \( \Delta \nu_D/\nu_o = v_{th}/c \lesssim 10^{-4} \), being able to actually resolve individual line profiles in a star’s spectrum demands a very high spectral resolution, \( R \equiv \nu/\Delta \nu \approx \lambda/\Delta \lambda \gtrsim 10,000 \), which requires some combination of a very bright star, a very large aperture telescope, or a very long exposure, to attain a sufficient number of photons within each frequency (or wavelength) resolution element. In lower-resolution survey spectra of stars, the individual line profiles are not discernible, and so the information about a given line is limited to some measure of its overall strength, meaning the total fraction of the continuum flux that has been reduced near the line.
Fig. 11.1.— Illustration of the definition of equivalent width $W_x$. The left panel plots the residual flux for a sample line (here with parameters $\beta_o = 100$, $\epsilon_c = 1$, $\epsilon_l = 0$, $b/a = 3/2$, and $a_v = 10^{-3}$), with the shaded area illustrating the total fractional reduction of continuum light. The right panel plots a box profile with width $W_x$, defined such that the total absorption area is the same for the curve to the left. In this example, $W_x = 3.01$.

This reduction can be characterized in terms of the equivalent width, defined mathematically as the frequency integral over the absorption fraction $A_\nu \equiv 1 - R_\nu$,

$$W_\nu \equiv \int_0^\infty A_\nu \, d\nu.$$  \hspace{1cm} (11.1)

As illustrated in figure 11.1, $W_\nu$ can be intuitively thought of as the width of a box profile with the same total flux reduction as the actual line.

11.2. Curve of Growth for Scattering and Absorption Lines

Given the residual flux $R_\nu$ and thus the absorption fraction $A_\nu$, the $W_\nu$ can be computed from direct (usually numerical) integration from eqn. (11.1). In practice, it is convenient to scale this width by the thermal Doppler width, $W_x \equiv W_\nu / \Delta \nu_D$, with the Doppler-scaled frequency $x = (\nu - \nu_o) / \Delta \nu_D$ used to define the frequency variation of line strength through $\beta_\nu = \beta_o \phi_v(x)$. Here $\phi_v(x)$ is the scaled Voigt profile, and $\beta_o$ a measure of line-strength at line-center, defined in terms of basic line parameters by

$$\beta_o = \sigma_{el} f_{ij} n_i \rho c \Delta \nu_D.$$  \hspace{1cm} (11.2)
Fig. 11.2.— Curves of growth for the case of pure-absorption continuum $\epsilon_c = 1$, with $b/a = 3/2$, and Voigt line damping parameters $a_v = 10^{-4}, 10^{-3},$ and $10^{-2}$. The left panel shows the pure absorption case given by eqn. (11.7), while the right panel is for the Milne-Eddington model for scattering lines with $\epsilon_l = 0$. The dashed lines compare linear (blue) and square root (red) functions, to show the limiting forms for very weak and very strong lines. The flatter portion bridging between these limits shows the “logarithmic” growth from intermediate strength lines with saturated Doppler cores, but no significant Lorentz damping wings.

This is proportional to the number of line absorbing atoms, and so the increase in of $W_x$ with increasing $\beta_0$, known as the curve of growth, represents how the integrated line attenuation from lines depends on the total number of absorbers.

The right panel of figure 11.2 shows such a theoretical curve of growth, computed from direct frequency integration of $A_\nu = 1 - R_\nu$, as derived from the above Milne-Eddington model of a pure scattering line with $\epsilon_l = 0$ in pure-absorbing continuum, with $\epsilon_c = 1$. The left panel shows corresponding curves of growth for a pure absorption line, again with absorption continuum, so that $\epsilon_\nu = \epsilon_c = \epsilon_l = 1$. Note that the basic forms of the curves of growth are quite similar in both the scattering and absorption cases, and indeed we will now see that this results mainly from the characteristics of the Voigt line profile.

To see this, it is helpful to focus on this relatively simple case of pure-absorption in both line and continuum, with again a linear Planck function $B(\tau) = a + b\tau_c$. But rather than use the above Milne-Eddington scalings that assume the Eddington approximation, in this pure-absorption case we can apply the moment form of the formal solution to compute
directly the emerging flux,

$$H_\nu(0) = \frac{1}{2} \int_0^\infty (a + b t) E_2[(1 + \beta_\nu)t](1 + \beta_\nu)dt$$

$$= \frac{1}{4} \left( a + \frac{b}{1 + \beta_\nu} \right).$$

(11.3)

From the ratio to the continuum, we obtain for the absorption strength

$$A_\nu = 1 - H_\nu(0)/H_c = \frac{\beta_\nu}{1 + \beta_\nu} \frac{1}{1 + 3a/2b} \equiv A_o \frac{\beta_\nu}{1 + \beta_\nu},$$

(11.5)

and thus the equivalent width

$$W_\nu = A_o \int_0^\infty \frac{\beta_\nu}{1 + \beta_\nu} d\nu = 2A_o \Delta \nu_D \beta_o \int_0^\infty \frac{\phi_v(x)}{1 + \beta_o \phi_v(x)} dx.$$

(11.6)

### 11.3. Linear, Logarithmic, and Square-Root Parts of the Curve of Growth

To isolate just this dependence on the number of absorbers through $\beta_o$, it is convenient to define a “reduced equivalent width”,

$$\frac{W_\nu}{2A_o \Delta \nu_D} \equiv W^*(\beta_o) = \beta_o \int_0^\infty \frac{\phi_v(x)}{1 + \beta_o \phi_v(x)} dx.$$

(11.7)

Note that this reduced equivalent width depends only on $\beta_o$ and the form of the (Voigt) profile function $\phi_v(x)$. Knowing that form, we can readily understand the various parts of the theoretical curves of growth in figure 11.2.

First, for weak lines with $\beta_o \lesssim 1$, we can generally ignore the $\beta_o$ term in the denominator of the integrand, implying that the integral just become the unit normalization of the profile function, and thus that the equivalent width just scales linearly with number of absorbers, $W^* \sim \beta_o$. Referring to the pure-absorption line profiles plotted in figure 10.1, this weak line regime corresponds to cases when the absorption is just starting to appear within the Doppler core of the line. Such relatively weak lines are said to lie on the “linear” part of the curve of growth.

But as we increase the number of absorbers, we soon come to a regime where this Doppler core becomes saturated, whereupon adding more absorbers hardly increases the total attenuation, since this requires expanding the width of the Doppler core against its strong Gaussian dependence on frequency. The increase in equivalent width for such intermediately strong lines is thus very slow, as $W^* \sim \sqrt{\log \beta_o}$, and so this regime is known as the “logarithmic” part of the curve of growth.
If, however, we increase $\beta_o$ still further, we now find that the line is becoming optically thick in the Lorentz damping wings, for which the line profile now scales only with the inverse square power of frequency displacement from line center, $\phi_v(x) \sim a_v/x^2$. As the increased number of absorbers extends the optical thick wings against this Lorentz scaling, the equivalent width approaches a scaling $W^* \sim \sqrt{\beta_o}$, and so such very strong lines with Lorentz damping wings represent a “square root” part of the curve of growth.

These separate domains are apparent for both the absorption (left) and scattering (right) line cases plotted in figure 11.2, but note that the transition from the logarithmic to square-root parts depends on the value of the Voigt damping parameter $a_v$.

**Exercise:**

a. For the case $a_v = 10^{-2}$, sketch a plot of $W^*$ vs. $\beta_o$ by estimating $W_*$ in limiting regimes appropriate to the “linear”, “logarithmic”, and “square root” parts of this curve of growth, marking the values for the transitions between these regimes.

b. Considering the definitions of $W^*$ and $\beta_o$, for each of these regimes discuss the dependencies of the actual equivalent width $W_\nu$ on the Doppler width $\Delta \nu_D$ and the number of absorbers $n_i$.

c. Based on your answer to (b), which part of an empirical curve of growth do you expect to be most appropriate for inferring the presence of “micro-turbulent” velocity fields? Which part for inferring elemental abundances? Explain the reasoning for your answers.

### 11.4. Doppler Broadening from Micro-turbulence

In our previous analysis of Doppler-broadening of a line profile, we assumed that the only speed to consider was that due to the random thermal motion of individual atoms, with an average thermal speed given in terms of the temperature $T$ and atomic mass $m$, $v_{th} = \sqrt{kT/m}$. But in practice, a stellar atmosphere is typically not completely static, but can include a spatially complex collection of eddies and swirls, for example associated with convective transport of energy from the interior, or from stellar pulsations, or from localized disturbances like flares. In lieu of developing a detailed model of a specific type of motion and its spatial character, it is often assumed that the motions can themselves be characterized in some simple statistical way, for example taking the fraction of atoms with a given turbulent speed $v$ to again scale as a Gaussian function, $e^{-(v/v_{turb})^2}$, where $v_{turb}$ now characterizes a root-mean-square measure of the turbulent velocity amplitude. Under the
further assumption that the spatial scale of such turbulence is much smaller than a typical photon mean-free-path, then the net effect would be add to the Doppler broadening of a line profile. Indeed, since the thermal motion of atoms and the random motion of such “microturbulence” are likely to be uncorrelated, the individual speeds should be added in quadrature, giving then a total average speed

\[ v_{\text{tot}} = \sqrt{v_{\text{th}}^2 + v_{\text{turb}}^2}, \]  

which then implies a total Doppler broadening

\[ \frac{\Delta \nu_D}{\nu_o} = \frac{\Delta \lambda_D}{\lambda_o} = \frac{v_{\text{tot}}}{c}, \]  

Since the reduced curve of growth is scaled by the Doppler width, matching an observed curve of growth can allow one to infer \( \Delta \nu_D \) or \( \Delta \lambda_D \). If \( v_{\text{th}} \) can then be inferred from the known stellar temperature e.g., \( T \approx T_{\text{eff}} \) and known atomic mass, then such a measure of the profile width can provide an estimate of the micro-turbulent velocity \( v_{\text{turb}} \). The details here are left as an exercise.

### 11.5. Rotational broadening of stellar spectral lines

In addition to the Doppler shift associated with the thermal or turbulent motion of gas on a small scale, there can be a large-scale Doppler shift from the parts of the star moving toward and away as a star rotates. This leads to a rotational broadening of the spectral lines, with the half-width in wavelength given by

\[ \frac{\Delta \lambda_{\text{rot}}}{\lambda_o} \equiv \frac{V_{\text{rot}} \sin i}{c}, \]  

where \( V_{\text{rot}} \) is the stellar surface rotation speed at the equator, and \( \sin i \) corrects for the inclination angle \( i \) of the rotation axis to our line of sight. If the star happens to be rotating about an axis pointed toward our line of sight \( (i = 0) \), then we see no rotational broadening of the lines. Clearly, the greatest broadening is when our line of sight is perpendicular to the star’s rotation axis \( (i = 90^o) \), implying \( \sin i = 1 \), and thus that \( V_{\text{rot}} = c \Delta \lambda_{\text{rot}}/\lambda_o \).

Figure 11.3 illustrates this rotational broadening. The left-side schematatic shows how a rotational broadened line profile for flux vs. wavelength takes on a hemi-spherical\(^{13} \) form.

\(^{13} \) If flux is normalized by the continuum flux \( F_c \), then making the plotted profile actually trace a hemisphere requires the wavelength to be scaled by \( \lambda_n \equiv \Delta \lambda_{\text{rot}}/r_o \), where \( \Delta \lambda_{\text{rot}} \) and \( r_o \) are the line’s rotational half-width and central depth, defined respectively by eqns. (11.10) and (11.12).
For a rigidly rotating star, the line-of-sight component of the surface rotational velocity just scales in proportion to the apparent displacement from the projected stellar rotation axis. Thus for an intrinsically narrow absorption line, the total amount of reduction in the observed flux at a given wavelength is just proportional to the area of the vertical strip with a line-of-sight velocity that Doppler-shifts line-absorption to that wavelength. As noted above, the total width of the profile is just twice the star’s projected equatorial rotation speed, $V \sin i$.

The right panel shows a collection of observed rotationally broadened absorption lines for a sample of quite rapidly rotating stars, i.e. with $V \sin i$ more than 100 km/s, much larger than the $\sim 1.8$ km/s rotation speed of the solar equator. The flux ratio here is relative to the nearby “continuum” outside the line.

Note that the reduction at line-center is typically only a few percent. This is because such rotational broadening preserves the total amount of reduced flux, meaning then that the relative depth of the reduction is diluted when a rapid apparent rotation significantly...
broadens the line. For a line with wavelength equivalent width,

\[ W_\lambda \equiv \int_0^\infty \left( 1 - \frac{F_\lambda}{F_c} \right) d\lambda, \]  

(11.11)

and a rotationally broadened half-width \( \Delta \lambda_{rot} \), the central reduction in flux is just

\[ r_o \equiv 1 - \frac{F_{\lambda_o}}{F_c} = \frac{2}{\pi} \frac{W_\lambda}{\Delta \lambda_{rot}}. \]  

(11.12)

**Exercise:** Derive eqn. (11.12) from the definitions of rotational Doppler width \( \Delta \lambda_{rot} \) (11.10) and equivalent width \( W_\lambda \) (11.11), using the wavelength scaling given in footnote 13.

For example, for the He 471.3 nm line plotted in the left, lowermost box in the right panel of figure 11.3, the central reduction is just \( r_o \approx 1 - 0.96 = 0.04 \), while the velocity half-width (e.g., given by the vertical red-dotted lines) is \( V \sin i \approx 275 \text{ km/s} \), corresponding to a wavelength half-width \( \Delta \lambda_{rot} \approx 0.43 \text{ nm} \). This implies an equivalent width \( W_\lambda \approx 0.027 \text{ nm} \), or about 17 km/s in velocity units.

### 12. NLTE Line Transfer

While very helpful for insight, the classical line transfer models of the previous sections are deficient in several respects, and so now let us develop a physically more realistic model, with emphasis on two particularly important improvements:

1. **CRD vs. Coherent Scattering.** Instead of assuming a “coherent” scattering in which each frequency remains isolated from all others, let us allow for frequency redistribution; specifically, let us instead make the opposite assumption to coherent scattering, namely that the photons undergo a randomization or Complete ReDistribution (CRD) in frequency within the line.

2. **Atomic Physics with Statistical Equilibrium.** Moreover, instead of assuming that the absorption and scattering opacities are given as fixed parameters – which then also set the line destruction probability \( \epsilon_l = \kappa_{abs}/(\kappa_{abs} + \kappa_{scat}) \) –, let us now compute these from basic atomic physics properties of absorbers, given also a solution to the ionization and excitation fractions of the ions based an general assumption of statistical equilibrium.
12.1. Two-Level Atom

A relatively simple example for this can be given by an approximate model of an atom that assumes it consists of just two bound energy states, representing an upper (u) and lower (l) level in energy. In terms of the Einstein coefficients for excitation from the lower level, and spontaneous and stimulated decay from the upper level, the equation of radiative transfer can be written as

\[ \mu \frac{dI_\nu}{dz} = \left[ -n_l B_{lu} + n_u (A_{ul} + B_{ul} I_\nu) \right] \frac{\phi_\nu h\nu}{4\pi}, \]  

(12.1)

where here we have effectively assumed CRD in invoking an equivalence between the emission and absorption profiles, \( \psi_\nu = \phi_\nu \). Let us next define a line-averaged opacity by

\[ \rho \kappa_{lu} = \left( n_l B_{lu} - n_u B_{ul} \right) \frac{h\nu}{4\pi \Delta \nu}, \]  

(12.2)

\[ = \sigma c f_{lu} \left( n_l - \frac{g_l}{g_u} n_u \right), \]  

(12.3)

where \( \Delta \nu \) is the line width, and the second term from stimulated emission now represents a kind of “negative absorption”\(^{14}\). To cast the transfer equation in its familiar form,

\[ \mu \frac{dI_\nu}{d\tau} = I_\nu - S_l, \]  

(12.4)

we now define the line source function by

\[ S_l \equiv \frac{n_u A_{ul}}{n_l B_{lu} - n_u B_{ul}} = \frac{2h\nu^3/c^2}{n_l g_u / n_u g_l - 1}, \]  

(12.5)

where the latter equality uses the Einstein relations. In the special condition of thermodynamic equilibrium (TE), the level populations follow the Boltzmann distribution,

\[ \frac{n_l g_u}{n_u g_l} = e^{h\nu/kT}, \]  

(12.6)

in which case the line source function reduces to the Planck function, \( S_l = B_{\nu_0} \).

However, in general conditions, \( S_l \) depends on the level populations \( n_l \) and \( n_u \), and these in turn depend on the radiation field. A more general condition for determining these populations is to assume a steady state in which the net processes creating and destroying

\(^{14}\)Indeed, in the case of a population inversion with \( n_l < n_u g_l / g_u \), this stimulated emission can dominate over positive absorption, producing a net amplification in intensity along a given beam. Such population inversions are the principle behind “Light Amplification by Stimulated Emission of Radiation”, or a laser.
each level must balance with the net processes creating and destroying the competing level. This balance is known as statistical equilibrium, and can be expressed by an equality between the total destruction rates from each of the two levels,

\[ n_l (B_{lu} \bar{J} + C_{lu}) = n_u (A_{ul} + B_{ul} \bar{J} + C_{ul}) \, . \] (12.7)

Here \( C_{lu} \) and \( C_{ul} \) represent collisional rates for excitation and de-excitation, and, under the assumption of CRD, the radiative rates now depend on the line-profile-averaged mean-intensity,

\[ \bar{J} = \int_{-\infty}^{\infty} \phi(x)J(x)dx \] (12.8)

Typically the free electrons that dominate the collisional rates have themselves a nearly Maxwell-Boltzmann distribution in energy, as in LTE. Thus, by a detailed balance argument, we require a further Einstein relation, now between the collisional excitation and de-excitation,

\[ C_{lu} = n_u^* C_{ul} = e^{-h\nu/kT} C_{ul} \, , \] (12.9)

where the asterisks denote the LTE populations. Applying this and the statistical equilibrium equation (12.7) in the line source function definition (12.5), we find

\[ S_l = \frac{\bar{J} + \epsilon' B_{\nu_0}}{1 + \epsilon'} = (1 - \epsilon) \bar{J} + \epsilon B_{\nu_0} \, , \] (12.10)

where the ratio of collisional to spontaneous decay is

\[ \epsilon' \equiv \frac{C_{ul}}{A_{ul}} (1 - e^{-h\nu/kT}) \, , \] (12.11)

with then the collisional destruction probability given by

\[ \epsilon \equiv \frac{\epsilon'}{1 + \epsilon'} \, . \] (12.12)

This is similar to the line destruction probability \( \epsilon_l \) defined above for classical line transfer, except that now it applies to the entire line, not just to a single frequency within the line. Otherwise, the meaning is quite similar, with \( \epsilon B_{\nu_0} \) representing the thermal creation of photons, and \( (1 - \epsilon) \bar{J} \) representing the scattering source.

But we now can readily see the essential physical scaling of this destruction probability, namely that since it depends the electron collisional rate \( C_{ul} \), it scales with the electron density, i.e., \( \epsilon \sim \epsilon' \sim C_{ul} \sim n_e \). This implies that at great depth, \( \epsilon' \to \infty \), and thus that \( \epsilon \to 1 \). Again, this leads the high-density regions at great depth toward LTE, \( \epsilon \approx 1 \).
whereas lower density regions in the atmosphere can have $\epsilon \ll 1$, implying a strong scattering component that characterizes NLTE.

But even in this strong scattering, NLTE regime, it is important realize that one cannot disregard the relatively small thermal term $\epsilon B_{\nu o}$, since it ultimately provides the source for creation of photons, which are then scattered within the atmosphere.

Indeed, if one sets $\epsilon = 0$, then note that the transfer equation becomes a homogeneous ordinary differential equation (ODE), with thus no scale for the radiation field, unless imposed externally through boundary conditions. In actual stellar atmospheres, this scale is indeed set by the small, $\epsilon B_{\nu o}$ thermal term, which when averaged over the volume of the thermalization depth, provide the ultimate source of the radiation.

However, we shall now see that the scaling for this thermalization depth is somewhat altered in the current case of CRD vs. the previous assumption of coherent scattering.

### 12.2. Thermalization for Two-Level Line-Transfer with CRD

For photons interacting in a spectral line, the destruction probability per encounter is given by

$$P_d = \frac{C_{ul}}{A_{ul} + C_{ul}} = \epsilon . \quad (12.13)$$

This is is to be compared with the probability for direct escape of a line photon with scaled frequency $x$ into direction cosine $\mu$ from a location with vertical optical depth at line center $\tau$,

$$P_e(x, \tau, \mu) = e^{-\tau \phi(x)/\mu} . \quad (12.14)$$

Averaged over the line profile $\phi(x)$ and all directions $\mu$,

$$P_e(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} dx \phi(x) \int_0^1 e^{-\tau \phi(x)/\mu} d\mu \quad (12.15)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} E_2[\tau \phi(x)]\phi(x) dx . \quad (12.16)$$

For strong lines with $\tau \gg 1$, let us define a line-edge frequency $x_1$ by $\phi(x_1) \equiv 1/\tau$. We then find that the exponential integral term in the integrand can be approximated by $E_2 \to 0$ for $x < x_1$, and by $E_2 \to 1$ for $x > x_1$. Physically this states that photons are trapped in the line core, with $x < x_1$, but can escape in the line wings, $x > x_1$. Thus the total escape probability can be approximated by

$$P_e(\tau) \approx \int_{x_1}^{\infty} \phi(x) dx . \quad (12.17)$$
The evaluation of this integral can be divided between the case of moderately strong line, for which $x_1$ is still in the Doppler core of the Voigt profile $\phi(x)$, and the case of very strong lines, for which $x_1$ lies in the Lorentz damping wings.

For intermediately strong lines with a Doppler profile, we find

$$P_e(\tau) = \frac{1}{2} \text{erfc}(x_1) \approx \frac{e^{-x_1^2}}{2\sqrt{\pi}x_1} = \frac{\phi(x_1)}{2x_1} = \frac{1}{2x_1\tau} \approx \frac{C}{\tau},$$

where in the last equality we note that $C = 1/2x_1 \approx 1/2\sqrt{\ln(\tau/\sqrt{\pi})}$ is nearly just a constant order unity.

If we then define a thermalization depth by setting the escape and destruction probabilities equal, $P_e(\tau_{th}) = P_d$, we find that for CRD line-transfer within the Doppler core the thermalization depth now scales as

$$\tau_{th} = \frac{C}{\epsilon}.$$

Recall that for coherent scattering, we found above that $\tau_{th} \sim 1/\sqrt{\epsilon}$. For the usual case that $\epsilon \ll 1$, the thermalization depth for CRD line-transfer is thus substantially deeper than in the coherent scattering case. For example, for a quite typical value $\epsilon \approx 10^{-4}$, we find that in CRD, $\tau_{th} \approx 10^4$, or a factor hundred greater than the $\tau_{th} \approx 100$ obtained by a coherent scattering model!

The physical reason for this difference stems from the difference in the nature of photon escape in coherent scattering vs. CRD. In coherent scattering, the photons can only escape through an extensive spatial diffusion, a random walk from their point of creation to the $\tau \approx 1$ layer for free photon flight and escape. By contrast, in CRD photons trapped in the very optically thick Doppler core of the line have a chance to be redistributed to a frequency in the line wings, where the optical depth can be of order unity or less, thus allowing a direct escape in a single flight, without the necessity of a protracted spatial diffusion. This greater “leakage” of thermally created photons means that the escape of radiation from the surface can be sensed to a much deeper level, implying then a much deeper thermalization depth.

**Exercise:** Show that for very strong lines with saturated Lorentz damping wings, the thermalization depth now scales as $\tau_{th} \sim 1/\epsilon^2$ if one assumes CRD holds through full Voigt profile out to the Lorentz wings.\(^{15}\) Give a physical explanation.

\(^{15}\)The assumption CRD actually falters for the Lorentz wings, which are better approximated by coherent scattering. As such, the actual thermalization depth for saturated line wings is not so deep in practice.
for this still deeper thermalization depth scaling.

13. The Energies and Wavelengths of Line Transitions

13.1. The Bohr Atom

The discretization of atomic energy that leads to spectral lines can be understood through the simple Bohr model of the Hydrogen atom. In analogy with planets orbiting the sun, this assumes that electrons of charge \(-e\) and mass \(m_e\) are in a stable circular orbit around the atomic nucleus (for hydrogen just a single proton) of charge \(+e\) whose mass is effectively infinite compared to the electron. The electrostatic attraction between these charges then balances the centrifugal force from the electron’s orbital speed \(v\) along a circular orbit of radius \(r\),

\[
\frac{e^2}{r^2} = \frac{m_e v^2}{r}.
\]  

(13.1)

In classical physics, this orbit could, much like a planet going around the sun, have any arbitrary radius. But in the microscopic world of atoms and electrons, such classical physics has to be modified – indeed replaced – by quantum mechanics. Just as a light wave has its energy quantized into discrete bundles called photons, it turns out that the orbital energy of an electron is also quantized into discrete levels, much like the steps of a staircase. The basic reason stems from the fact that, in the ghostly world of quantum mechanics, electrons are themselves not entirely discrete particles, but rather, much like light, can also have a “wavelike” character. In fact any particle with momentum \(p = mv\) has an associated “\(de\) Broglie wavelength” given by

\[
\lambda = \frac{h}{mv},
\]

(13.2)

where again, \(h\) is Planck’s constant.

This wavy fuzziness means an orbiting electron cannot be placed at any precise location, but is somewhat spread along the orbit. But then to avoid “interfering with itself”, integer multiples \(n\) of this wavelength should match the orbital circumference \(2\pi r\), implying

\[
n\lambda = 2\pi r = \frac{nh}{mv}.
\]

(13.3)

Note that Planck’s constant itself has units of speed times distance \(^{16}\), which represents

\(^{16}\)Or also, energy times time, which when used with Heisenberg’s Uncertainty \(\Delta E \Delta t \gtrsim h\), will lead us to conclude that an atomic state with finite lifetime \(t_{life}\) must have a finite width or “fuzziness” in its energy \(\Delta E \sim h/t_{life}\). See the section below on “natural broadening” of spectral lines.
an angular momentum. So another way to view this is that the electron’s orbital angular momentum \( J = mvr \) must likewise be quantized,

\[
J = mvr = n\hbar ,
\]

where the “reduced Planck constant” \( \hbar \equiv h/2\pi \) is a standard notation shortcut. The integer index \( n \) is known as the principal quantum number.

**Exercise 2-1:** Use eqns. (13.1) and (13.4) to derive the orbital radius \( r_n \) in terms of the integer step \( n \).

**Exercise 2-2:** For an electron and proton that are initially a distance \( r \) apart, show that the energy needed to separate them to an arbitrarily large distance is given by \(-U(r) = e^2/r\). Use the resulting potential energy \( U(r) \) together with the orbital kinetic energy \( T = m_e v^2/2 \) to derive the expressions in eqn. (13.5) for the total energy \( E = U + T \).

The quantization condition in eqn. (13.3) or (13.4) implies that the orbital radius can only take certain discrete values \( r_n \), numbered by the level \( n \). But instead of radius, it is generally more useful to cast this in terms of the associated orbital energy. The total orbital energy is a combination of the negative potential energy \( U = -e^2/r \), and the positive kinetic energy \( T = m_e v^2/2 \). Using the orbital force balance eqn. (13.1), we find that the total energy is

\[
E_n = -\frac{e^2}{2r_n} = -\frac{m_e e^4}{2\hbar^2} \frac{1}{n^2} = -\frac{E_1}{n^2} ,
\]

where

\[
E_1 = \frac{m_e e^4}{2\hbar^2} = 2.2 \times 10^{-11} \text{erg} = 13.6 \text{eV}
\]

denotes the ionization (a.k.a. binding) energy of Hydrogen from the ground state (with \( n = 1 \)). Figure 13.1 gives a schematic rendition of the energy levels of Hydrogen, measured in electron Volts (eV), which is the energy gained when a charge of one electron falls through an electrical potential of one volt.

**Exercise 2-3:** Confirm the validity of eqn. (13.5) by using eqn. (13.1) to show that \( E = U/2 = -T \), where \( U, T, E \) are the potential, kinetic, and total energy of an orbiting electron. (Note: this result is sometimes referred to as a corollary of the Virial Theorum for bound systems, which is discussed further below.)
When an electron changes from one level with quantum number \( m \) to another with quantum number \( n \), then the associated change in energy is

\[
\Delta E_{mn} = E_1 \left( \frac{1}{n^2} - \frac{1}{m^2} \right).
\] (13.7)

### 13.2. Line Wavelengths for Term Series

Instead of energy, light is more commonly measured in terms of its wavelength \( \lambda = c/\nu = hc/E. \) Using this conversion in eqn. (13.7), we find the wavelength of a photon emitted by transition from a level \( m \) to a lower level \( n \) is

\[
\lambda_{mn} = \frac{\lambda_1}{n^2 - m^2},
\] (13.8)
where
\[ \lambda_1 \equiv \frac{hc}{E_1} = \frac{\hbar^3 c}{2\pi^2 m_e e^4} = 91.2 \text{ nm} = 912 \text{ Å} \] (13.9)
is the wavelength at what is known as the Lyman limit, corresponding to a transition to the ground state \( n = 1 \) from an arbitrarily high bound level with \( m \to \infty \). Of course, transitions from a lower level \( m \) to a higher level \( n \) require absorption of a photon, with the wavelength now given by the absolute value of eqn. (13.8).

The lower level of a transition defines a series of line wavelengths for transitions from all higher levels. For example, the Lyman series represents all transitions to/from the ground state \( n = 1 \). Within each series, the transitions are denoted in sequence by a lower case greek letter, e.g. \( \lambda_{21} = (4/3) \times 912 = 1216 \text{ Å} \) is called Lyman-\( \alpha \), while \( \lambda_{31} = (9/8)912 = 1026 \text{ Å} \) is called Lyman-\( \beta \), etc. The Lyman series all falls in the ultraviolet (UV) part of the spectrum, which due to UV absorption by the earth’s atmosphere is generally not possible to observe from ground-based observatories.

More accessible is the Balmer series, for transitions between \( n = 2 \) and higher levels with \( m = 3, 4, \) etc., which are conventionally denoted \( \text{H}\alpha, \text{H}\beta, \) etc. These transitions are pretty well positioned in the middle of the visible, ranging from \( \lambda_{32} = 6566 \text{ Å} \) for \( \text{H}\alpha \) to \( \lambda_{\infty 2} = 3648 \text{ Å} \) for the Balmer limit.

The Paschen series, with lower level \( n = 3 \), is generally in the InfraRed (IR) part of the spectrum. Still higher series are at even longer wavelengths.

### 14. Equilibrium Excitation and Ionization Balance

#### 14.1. Boltzmann equation

A key issue for formation of spectral lines from bound-bound transitions is the balance of processes that excite and de-excite the various energy levels of the atoms. In addition to the photon absorption and emission processes discussed above, atoms can also be excited or de-excited by collisions with other atoms. Since the rate and energy of collisions depends on the gas temperature, the shuffling among the different energy levels also depends sensitively on the temperature.

In Thermodynamic Equilibrium (TE), the numbers in each level \( i \) is just proportional to the number of quantum mechanical states, \( g_i \), associated with the orbital and spin state
of the electrons in that level\(^{17}\); but between a lower level \(i\) and upper level \(j\) with an energy difference \(\Delta E_{ij}\), the relative population is also weighted by an exponential term called the Boltzmann factor,

\[
\frac{n_j}{n_i} = \frac{g_j}{g_i} e^{-\Delta E_{ij}/kT},
\]

where \(k = 1.38 \times 10^{-16} \text{ erg/K}\) is known as Boltzmann’s constant. At low temperature, with the thermal energy much less than the energy difference, \(kT \ll \Delta E_{ij}\), there are relatively very few atoms in the more excited level \(j\), \(n_j/n_i \to 0\). Conversely, at very high temperature, with the thermal energy much greater than the energy difference, \(kT \gg \Delta E_{ij}\), the ratio just becomes set by the statistical weights, \(n_j/n_i \to g_j/g_i\).

As the population in excited levels increases with increased temperature, there are thus more and more atoms able to emit photons, once these excited states spontaneously decay to some lower level. This leads to an increased emission of the associated line transitions.

On the other hand, at lower temperature, the population balance shifts to lower levels. So when these cool atoms are illuminated by continuum light from hot layers, there is a net absorption of photons at the relevant line wavelengths, leading to a line-absorption spectrum.

### 14.2. Saha Equation for Ionization Equilibrium

At high temperatures, the energy of collisions can become sufficient to overcome the full binding energy of the atom, allowing the electron to become free, and thus making the atom an ion, with a net positive charge. For atoms with more than a single proton, this process of ionization can continue through multiple stages up to the number of protons, at which point it is completely stripped of electrons. Between an ionization stage \(i\) and the next ionization stage \(i + 1\), the exchange for any element \(X\) can be written as

\[
X_{i+1} \leftrightarrow X_i + e^-.
\]

In thermodynamic equilibrium, there develops a statistical balance between the neighboring ionization stages that is quite analogous to the Boltzmann equilibrium for bound levels given in eqn. (14.1). But now the ionized states consist of both ions, with many discrete energy levels, and free electrons, with a kinetic energy \(E = \frac{p^2}{2m_e}\) given by their momentum \(p\) and mass \(m_e\). The number of bound states of an ion in ionization stage \(i\) is

\(^{17}\)These orbital and spin states are denoted by quantum mechanical number \(\ell\) and \(m\), which thus supplement the principal quantum number \(n\).
now given by something called the partition function, which we will again write as \( g_i \). But to write the equilibrium balance, we now need also to find an expression for the number of states available to the free electron.

For ionization of a stage \( i \) with ionization energy \( \Delta E_i \), the Boltzmann relation for the ratio of upper vs. lower ionization state can then be written

\[
\frac{n_{i+1}(p)}{n_i} = \frac{g_{i+1}}{g_i} g_e(p) e^{-\left(\Delta E_i + p^2/2m_e\right)/kT}, \tag{14.3}
\]

where \( n_{i+1}(p) \, dp \) is the number of ionized atoms with an associated electron of momentum between \( p \) and \( p + dp \), and \( g_e(p) \) is the statistical weight for such electrons, representing the number of quantum mechanical states available to them. Because electrons with momentum \( p \) have an associated de Broglie wavelength \( \lambda_p = h/p \), each electron occupies a minimum volume \( h^3 \) in “phase space”, with dimensions of length times times momentum. For an isotropic momentum distribution, the momentum volume is that for shell of radius \( p \) and thickness \( dp \), i.e. \( 4\pi p^2 \, dp \), while the spatial volume is just the inverse of the total density. Accounting for the two possible states of the electron spin, we then find the number of available states is

\[
g_e(p) \, dp = \frac{2}{n_e} \frac{4\pi p^2 \, dp}{h^3}. \tag{14.4}
\]

If we define a de Broglie wavelength associated with thermal-speed electrons,

\[
\Lambda \equiv \frac{h}{\sqrt{2\pi m_e kT}}, \tag{14.5}
\]

then after integrating eqn. (14.3) over all \( p \) from zero to infinity, we obtain the Saha-Boltzmann equation for ionization balance

\[
\frac{n_{i+1}}{n_i} = \frac{g_{i+1}}{g_i} \left[ \frac{2}{n_e \Lambda^3} \right] e^{-\Delta E_i/kT} \tag{14.6}
\]

\[
\left[ \frac{n_{i+1}}{n_i} \right] = \frac{g_{i+1}}{g_i} \frac{2}{n_e} \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\Delta E_i/kT}. \tag{14.7}
\]

Throughout a normal star, the electron state factor in square brackets is typically a huge number\(^{18} \). For example, for conditions in a stellar atmosphere, it is typically of order \( 10^8 \). This large number of states acts like a kind of “attractor” for the ionized state. It means

\(^{18}\text{As discussed later, it only becomes order unity in very compressed conditions, like in the interior of a white dwarf star, which is thus said to be electron degenerate.}\)
the numbers in the more vs. less ionized states can be comparable even when the exponential Boltzmann factor is very small, with a thermal energy that is well below the ionization energy, i.e. $kT \approx \Delta E_i/10$.

For example, hydrogen in a stellar atmosphere typically starts to become ionized at a temperature of about $T \approx 10^4 K$, even though the thermal energy is only $kT \approx 0.86 \text{ eV}$, and thus much less than the hydrogen ionization energy $E_i = 13.6 \text{ eV}$. But this leads to a Boltzmann factor $e^{-13.6/0.86} = 1.4 \times 10^{-7}$ that is roughly offset by this large electron states factor. If we assume the partition ratio is of order unity, this gives a roughly equal fraction of Hydrogen in neutral and ionized states.

14.3. Absorption Lines and Spectral Type

![stellar spectra](image)

Fig. 14.1.— Stellar spectra for the full range of spectral types OBAFGKM, corresponding to a range in stellar surface temperature from hot to cool.

The line-absorption patterns that appear in a star’s spectrum depend on the elemental composition and degree of ionization of the atoms. Because ionization depends sensitively on temperature, the lines in stars with higher (lower) surface temperatures generally come from higher (lower) ionization stages. The effect is so systematic that a judicious classification of the spectral lines from a star can be used to infer the surface temperature. Figure 14.1 compares the spectra of stars of different surface temperature, showing that this leads to gradual
changes and shifts in the detailed pattern of absorption lines. The letters “OBAFGKM” represent various categories, known as “spectral class”, assigned to stars with different spectral patterns. It turns out that class O is the hottest, with temperatures about 50,000 K, while M is the coolest, with temperatures of ca. 3500 K. The sequence is often remembered through the mnemonic “Oh, Be A Fine Gal/Guy Kiss Me”.

Figure 14.2 shows some further spectra, now broken down into subtypes denoted by an added number from 0 to 9, representing a further delineation of hot to cool within the main
spectral type. For example, the sun is a G3 star, with a spectrum somewhat intermediate between the slightly hotter G0 and slightly cooler G5 types shown in the figure. The labels along the top and bottom now also identify the specific elements responsible for the most prominent absorption lines. Note for the high temperature stars, the labels along the top generally correspond to atoms or ions, whereas for cooler stars the labels along the bottom are often for molecules. At high temperature, the higher energy of both the photons and the collisions between the atoms is sufficient to strip off the electrons from atoms, whereas at low temperatures, the energy of the photons and collisions is low enough to allow the much weaker bonds of molecules to still survive.

14. Luminosity class

The Saha equation shows that ionization depends on density as well as temperature, and for a given temperature, density depends on pressure. Through the equation of hydrostatic equilibrium, the pressure in an atmosphere can be written as $P = mg = \tau g/\kappa$, where the mass column density $m$ is related to the optical depth through the opacity $\kappa$ by $m = \tau/\kappa$. Thus at the photospheric layers $\tau = 1$, we see that the pressure scales directly with the stellar gravity, which, for given stellar mass, depends in turn on the stellar radius, $g = GM/R^2$. For a given surface temperature, a larger radius implies a larger surface area and so a larger luminosity, $L \sim \sigma T^4 4\pi R^2$.

The upshot then is that an ionization balance that suggests a lower density – and thus lower pressure, lower gravity, larger radius – also implies a higher luminosity. This is the basis of the luminosity class of stellar spectra, conventionally denoted with roman numerals I, II, III, IV and V, denoting a declining luminosity sequence. Class I are called supergiants, representing very large, and very luminous stars. Class III are just ordinary giants, still large and luminous, but less so. Finally class V are dwarfs, representing “normal” stars like the sun.

15. H-R Diagram: Color-Magnitude or Temperature-Luminosity

15.1. H-R Diagram for Stars in Solar Neighborhood

A key diagnostic of stars comes from the Hertzsprung-Russel (H-R) diagram. Observationally, it relates stellar colors to their (absolute) magnitude, or spectral type and luminosity class; physically, it relates surface temperature to luminosity. Figure 14.3 shows an H-R diagram for a large sample of stars with known luminosities and colors. The horizontal lines
Fig. 14.3.— H-R diagram relating stellar luminosity to surface temperature. The points include 22,000 stars from the Hipparcos Catalogue together with 1000 low-luminosity stars (red and white dwarfs) from the Gliese Catalogue of Nearby Stars.

show the luminosity classes.

The extended band of stars running from the upper left to lower right is known as the main sequence, representing “dwarf” stars of luminosity class V. The reason there are so many stars in this main sequence band is that it represents the long-lived phase when stars are stably burning Hydrogen into Helium in their cores.

The medium horizontal band above the main sequence represent “giant stars” of lumi-
nosity class III. They are typically stars that have exhausted hydrogen in their core, and are now getting energy from a combination of hydrogen burning in circum-core shells, and burning Helium into Carbon in their cores.

The relative lack here of still more luminous supergiant stars of luminosity class I stems from both the relative rarity of stars with sufficiently high mass to become this luminous, coupled with the fact that such luminous stars only live for a very short time. As such, there are only a few such massive, luminous stars in the solar neighborhood. Studying them requires broader surveys that encompass a greater fraction of the galaxy.

The band of stars below the main sequence are called white dwarfs; they represent the slowly cooling remnant cores of low-mass stars like the sun.

This association between position on the H-R diagram, and stellar parameters and evolutionary status, represents a key link between the observable properties from stellar atmosphere and the physical properties associated with the stellar interior. Understanding this link through examination of stellar structure and evolution will constitute the major thrust of our studies of stellar interiors below.

Fig. 15.1.— Left: H-R diagram for globular cluster M55, showing how stars on the upper main sequence have evolved to lower temperature giant stars. Right: Schematic H-R diagram for clusters, showing the systematic peeling off of the main sequence with increasing cluster age.
15.2. H-R Diagram for Clusters – Evolution of Upper Main Sequence

The above volume-limited sample near the sun consists of stars of a wide range of ages, distances, and perhaps even chemical composition. But stars in a stellar cluster are all at a similar distance, and since the likely formed over a relatively short time span out of the same interstellar cloud, they should all have nearly the same age and composition. The left panel of figure 15.1 plots an H-R diagram for the globular cluster M55. Note that all the stars in the upper left main sequence have evolved to a vertical branch of cooler stars extending up to the red giants. This reflects the fact that more luminous stars exhaust their hydrogen fuel sooner than dimmer stars. The right panel illustrates this schematically, showing how the turnoff point from the main sequence is an indicator of the cluster age. Plots like this thus provide a direct diagnostic of stellar evolution of stars with different luminosity. As we shall now see, the main sequence luminosity of stars is set primarily by the stellar mass.

16. The Mass-Luminosity Relation for Main Sequence Stars

16.1. Stellar Masses Inferred from Binary Systems

The most direct and robust way to infer stellar masses comes from using Kepler’s laws to interpret their inferred motion in binary systems, with the most accurate masses derived from astrometric binaries and double line eclipsing binaries.

In astrometric binaries, the positions of the stars can be directly measured as they orbit their common center-of-mass (CM), and since the CM lies at a focus of the elliptical orbit, the inclination of the orbit can be inferred. For a given inclination, measuring the Doppler shift of spectral lines of each star (if bright enough) gives their orbital velocities, which combined with the measured orbital period gives the absolute dimensions of the orbits, the semi-major axes. Through Kepler’s third law the orbital period and semi-major axes of each star gives the mass of each component. Furthermore, a comparison of the absolute dimensions with the apparent angular size gives the distance to the binary, allowing apparent magnitudes to be converted to absolute magnitudes and hence luminosities.

In double-line eclipsing binaries, the inclination of the orbit is obtained from the eclipse light curve. The masses can then be obtained in the same way as for the astrometric binaries. If the distance is not known from trigonometric parallax, a distance estimate can be obtained from the eclipse durations, which give the sizes of the eclipsing stars. If the temperatures of the stars can be obtained from their spectra, combining with the stellar radii gives the stellar luminosity. The distance is then obtained by comparing with the apparent magnitudes.
Fig. 16.1.— A log-log plot of luminosity vs. mass (in solar units) for a sample of astrometric (blue, lower points) binaries and eclipsing (red, upper points) binaries. The best-fit line shown follows the empirical scaling, $\log \left( \frac{L}{L_\odot} \right) \approx 0.1 + 3.1 \log \left( \frac{M}{M_\odot} \right)$.

Figure 16.1 plots $\log L$ vs. $\log M$ (in solar units) for a sample of astrometric (blue) and eclipsing (red) binaries. A key result is that the data can be roughly fit by a straight line in this log-log plot, implying a power-law relation between luminosity and mass,

$$\frac{L}{L_\odot} \approx \left( \frac{M}{M_\odot} \right)^{3.1}. \quad \text{(16.1)}$$

16.2. Simple Theoretical Scaling Law for Mass vs. Luminosity

This empirical mass-luminosity relation can roughly explained by considering two basic relations of stellar structure, namely hydrostatic equilibrium and radiative diffusion, as given in eqns. (3.3) and (7.18) above. As in the virial scaling for internal temperature given in section 3.1, we can use a single point evaluation of the pressure gradient to derive a scaling
between interior temperature $T$, stellar radius $R$ and mass $M$, and molecular weight $\mu$,

\[
\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}
\]

\[
\frac{T}{\rho \mu R} \sim \frac{M}{R^2}
\]

\[
TR \sim M \mu,
\]

Likewise, a single point evaluation of the temperature gradient in the radiative diffusion equation gives,

\[
F(r) = -\left[\frac{4ac}{3} T^3 \frac{T}{3\kappa R \rho}\right] \frac{dT}{dr}
\]

\[
\frac{L}{R^2} \sim \frac{R^3 T^4}{\kappa M R}
\]

\[
L \sim \frac{(RT)^4}{\kappa M}
\]

\[
L \sim \frac{M^3 \mu^4}{\kappa},
\]

where the last scaling uses the hydrostatic equilibrium result to derive the basic scaling law $L \sim M^3$, assuming a fixed molecular weight $\mu$ and opacity $\kappa$.

We thus see that the empirical scaling found from binary systems plotted in figure 16.1 can be understood just in terms of the two basic equations for the structure of the stellar envelope, namely hydrostatic balance against gravity, and radiative diffusion transport of the stellar luminosity outward.

Note in particular that it does not depend on the details of the nuclear generation of the luminosity in the stellar core! Indeed, this scaling was understood from stellar structure analyses that were done (e.g. by Eddington, and Schwarzschild) in the 1920’s, long before nuclear burning was firmly established.

It is also completely independent of the stellar radius $R$, which cancels in the above scaling for luminosity.
16.3. Virial Theorem

The hydrostatic balance of a star can be used to derive a relation – known as the virial theorem – between the internal thermal energy and the gravitational binding energy of a star. Detailed derivations are given in Rich Townsend’s notes “01-virial.pdf” and Jim MacDonald’s “Notes_Part_3.pdf”, so here we just give an abridged derivation. Let us first multiply the standard hydrostatic equilibrium equation (3.3) by the radius \( r \), and integrate over a mass coordinate \( dm = 4\pi r^2 \rho dr \), with \( m \) ranging from 0 to \( M \) as \( r \) goes from 0 to \( R \),

\[
- \int_0^M \frac{Gm}{r} \, dm = \int_0^M r \frac{dP}{\rho} \, dm = \int_0^M 4\pi r^3 \frac{dP}{dm} \, dm = [4\pi r^3 P(r)]^{r=R}_{r=0} - \int_0^M \frac{P}{\rho} \, dm = -3(\gamma - 1) \int_0^M u \, dm
\]

\( \Phi = -3(\gamma - 1)U. \) \hfill (16.5)

On the left side, \(-Gm/r\) is the gravitational potential, defined such that zero potential is at \( r \to \infty \); the integrand thus gives the gravitational energy gained by adding each mass shell \( dm \), so that the total integral represents the total gravitational energy \( \Phi \). On the right side, integration by parts leads to the square bracket term, which vanishes at both the center, \( r = 0 \), and at surface, where the gas pressure effectively becomes negligible, \( P(R) = 0 \). For the remaining term, we note that for a perfect gas, \( P/\rho = (\gamma - 1)u \), where \( u \) is the internal energy-per-unit mass. The integration over mass thus gives the right side in terms of the total internal energy of the star, \( U \). For monotonic gas with \( \gamma = 5/3 \), we thus find

\( \Phi = -3(\gamma - 1)U = -2U, \) \hfill (16.6)

This implies that the total stellar energy is given by

\[ E \equiv \Phi + U = \frac{3\gamma - 4}{3\gamma - 3} \Phi = \frac{\Phi}{2} < 0, \] \hfill (16.7)

where again the last equality is for the case of a monotonic gas case \( \gamma = 5/3 \), which is indeed the most appropriate for stellar interiors.

The virial theorem thus implies that the total energy of a star is negative (meaning it is bound), with a numerical value just half of the gravitational binding energy. This reduction by one-half stems from the positive internal energy, which equals half the absolute value of the gravitational energy.
However, in very massive stars the internal energy (and pressure) can become dominated by radiation instead of gas. In such a radiation dominated gas, \( \gamma \rightarrow 4/3 \), which by eqn. (16.7) implies a total energy \( E \rightarrow 0 \). This weak to null level of gravitational binding implies that a radiation-dominated stellar envelope can become disrupted, possibly ejecting large amounts of mass in intervals when the star’s radiation force exceeds the force of gravity. This is closely linked to the Eddington limit, and as noted in §19.4, may be a key factor in setting an upper mass limit of stars, on the order of a few hundred \( M_\odot \).

17. Characteristic Timescales

17.1. Shortness of Chemical Burning Timescale for Sun and Stars

When 19th century scientists pondered the possible energy sources for the sun, some first considered whether this could come from the kind of chemical reactions that provide a key energy source (e.g. from fossil fuels like coal) on earth. But such chemical reactions involve transitions of electrons among various bound states of atoms, and, as discussed in the above Bohr model of the Hydrogen, the scale of energy release in such transitions is limited to about an order of electron volt (eV). In contrast, the rest mass energy of the atom itself, consisting protons and neutrons, is typically of order 10 Gev, or \( 10^{10} \) times higher. With the associated mass-energy efficiency of \( \epsilon \sim 10^{-10} \), we can readily estimate a timescale for maintaining the solar luminosity from chemical reactions,

\[
t_{\text{chem}} = \epsilon \frac{M_\odot c^2}{L_\odot} = \epsilon \times 4.5 \times 10^{20} \text{ sec} = \epsilon \times 1.5 \times 10^{13} \text{ yr} \approx 15,000 \text{ yr} .
\] (17.1)

Even in the 19th century, it was clear, e.g. from geology processes like erosion, that the earth – and so presumably also the sun – had to be much older than this.

17.2. Kelvin-Helmholtz Timescale for Luminosity Powered by Gravity

So let us instead consider a timescale associated with gravitational contraction as the energy source. The above virial relation shows that, as a star undergoes a gradual contraction that roughly maintains hydrostatic equilibrium, half of the gained energy goes into internal energy \( U \) of the star. But then the other half is available to be radiated away, powering the stellar luminosity. Following the work by Kelvin and Helmholtz, we can thus define an associated gravitational contraction lifetime for the sun

\[
t_{KH} \equiv -\frac{\Phi}{L} .
\] (17.2)
To estimate a value for the gravitational binding energy, let us first consider the somewhat artificial assumption that the sun has a uniform density, given by its mass over volume, \( \rho = M_\odot/(4\pi R_\odot^3/3) \). Then the equation for gravitational binding energy gives

\[
-\Phi = \int_0^{M_\odot} \frac{Gm}{r} \, dm = \frac{16\pi^2}{3} G\rho^2 \int_0^R r^4 \, dr = \frac{3GM_\odot^2}{5R_\odot}. \tag{17.3}
\]

Applying this in eqn. (17.2), we find

\[
t_{KH} \approx \frac{3}{5} \frac{GM_\odot^2}{R_\odot L_\odot} \approx 30 \text{ Myr}. \tag{17.4}
\]

Although substantially longer than the chemical burning timescale, this is still much shorter than the geologically inferred minimum age of the earth, which is at least a Gyr.

### 17.3. Nuclear Burning Timescale

We now realize, of course, that the main sequence age of stars like the sun is set by a much longer nuclear burning timescale. When four hydrogen nuclei are fused into a helium nucleus, the helium mass is about 0.7% lower than the original four hydrogen. For nuclear fusion the above-defined mass-energy burning efficiency is thus \( \epsilon \approx 0.007 \). But in typical main sequence star, only some core fraction \( f = 1/10 \) of the stellar mass ever become hot enough to allow hydrogen fusion. Applying this we thus find for the nuclear burning timescale

\[
t_{nuc} = \epsilon_{nuc} f \frac{M c^2}{L} = 0.007 \times 0.1 \times 1.5 \times 10^{13} \text{ yr} = 10^{10} \text{ yr}, \tag{17.5}
\]

where the latter equality applies the solar values. This is the basic rationale for the above quote (e.g. eqn. (17.6)) that the sun’s main sequence lifetime is about 10 Gyr.

### 17.4. Main Sequence Lifetimes and the Age of Clusters from MS Turnoff

The above scaling of main-sequence luminosity with a high power of the stellar mass provides a simple explanation for the progressively truncated form of the upper main-sequence in older stellar clusters. Let us make the reasonable assumption that a fixed fraction of the total hydrogen mass of any star is available for nuclear burning into helium in its stellar core. The fuel available then scales with the mass, but the burning rate depends on the luminosity. Normalized to the sun, the main-sequence lifetime thus scales as

\[
t_{ms} = t_{ms,\odot} \frac{M/M_\odot}{L/L_\odot} \approx 10 \text{ Gyr} \left( \frac{M_\odot}{M} \right)^2. \tag{17.6}
\]
The most massive stars, of order 100 $M_\odot$, thus have main-sequence lifetimes about about 1 Myr, much shorter the multi-Gyr timescale for solar mass stars.

For cluster H-R diagrams, the above scalings mean that luminosity of stars at the turn-off point of the main sequence, $L_{to}$, can be used to infer the cluster age,

$$t_{\text{cluster}} \approx 10 \text{ Gyr} \left( \frac{L_\odot}{L_{to}} \right)^{2/3}.$$  \hspace{1cm} (17.7)

18. Structure of Radiative vs. Convective Stellar Envelopes

18.1. Convective instability and energy transport

In practice, the transport of energy from the stellar interior toward the surface sometimes occurs through convection instead of radiative diffusion, and this has important consequence for stellar structure and thus for the scaling of luminosity.

Convection refers to the overturning motions of the gas, much like the bubbling of boiling water on a stove. Stars become unstable to forming convection whenever the processes controlling the temperature make its spatial gradient too steep. This can occur in the nuclear burning core of massive stars, for which the specific mechanism for H-fusion, called the “CNO” cycle, gives the nuclear burning rate a steep dependence on temperature. The resulting steep temperature gradient makes the cores of such stars strongly convective.

Steep gradients, and their associated convection, can also occur in outer regions cooler, lower-mass stars, where the cooler temperature induces recombination of ionized H or He. The bound-free absorption by this neutral Hydrogen significantly increases the local stellar opacity $\kappa$. For a fixed stellar flux $F = L/4\pi r^2$ of stellar luminosity $L$ that needs to be transported through an interior radius $r$, the radiative diffusion eqn. (7.18) shows that the required radiative temperature gradient increases with such increased opacity,

$$\left| \frac{dT}{dr} \right|_{\text{rad}} = \frac{3\kappa \rho F}{16\sigma_{sb}T^3} \sim \kappa$$  \hspace{1cm} (18.1)

If this gradient becomes too steep, then, as illustrated in figure 18.1, a small element of gas that is displaced slightly upward becomes less dense than its surroundings, giving it a buoyancy that causes it to rise higher still. A key assumption is that this dynamical rise of the fluid occurs much more rapidly than the rate for energy to diffuse into or out of the gas element. Processes that occur without any such energy exchange with the surroundings
Fig. 17.1.— Illustration of the pre-main-sequence evolution of stars. The upper panel shows how during the early stages of a collapsing proto-star, the interior is fully convective, causing it to evolve with decreasing luminosity at a nearly constant, relatively cool surface temperature, and so down the nearly vertical “Hayashi track” in the H-R diagram. The lower panel shows the final approach to the main sequence for stars of various masses. For stars with a solar mass or above, the stellar interior becomes radiative, stopping the Hayashi track decline in luminosity. The stars then evolve horizontally and to the left on the H-R diagram, each with fixed luminosity but increasing temperature, till they reach their respective positions on the “zero-age-main-sequence” or ZAMS, when the core is hot enough to ignite H-fusion.
Fig. 18.1.— Illustration of upward displacement of a spherical fluid element in test for convective instability, which occurs when the displaced element has a lower density $\rho'_1$ than that of its surroundings, $\rho_1$. Since the pressure must remain equal inside and outside the element, this requires the element to have a higher temperature, $T'_1 > T_1$. Since the overall temperature gradient is negative, convection thus occurs whenever the magnitude of the atmospheric temperature gradient is steeper than the adiabatic gradient that applies for the adiabatically displaced element, i.e., $|dT/dr| > |dT/dr|_{ad}$.

are called “adiabatic”, with a fixed (power-law) relation of pressure with density or temperature. In a hydrostatic medium with a set pressure gradient, this implies a fixed adiabatic temperature gradient $(dT/dr)_{ad}$.

Starting from an initial radius $r_0$ with equal density and temperature inside and outside some chosen fluid element (i.e., $\rho'_0 = \rho, T'_0 = T_0$), let us determine the density $\rho'_1$ of that element after it is adiabatically displaced to a slightly higher radius $r_1 = r_0 + \delta r$, where the ambient density is $\rho_1$. Since dynamical balance requires the element and its surrounding to still have equal pressure after the displacement (i.e., $P'_1 = P_1$), we have by the perfect gas law that $\rho'_1 T'_1 = \rho_1 T_1$. If this upward displacement $\delta r > 0$ makes the element buoyant, with lower density $\rho'_1$ than that of its surroundings $\rho_1$, then using this constant pressure condition, we can derive the condition for the temperature gradient required for the associated convective
instability,

\[ \frac{T_1}{T_1'} = \frac{\rho'}{\rho} < 1 \quad ; \quad \text{Convective instability} \]

\[ T_0 + \Delta r (dT/dr)_{rad} = T_1 < T_1' = T_0 + \Delta r (dT/dr)_{ad} \]

\[ \left| \frac{dT}{dr} \right|_{rad} > \left| \frac{dT}{dr} \right|_{ad} \]

\[ \nabla_{rad} > \nabla_{ad}, \quad (18.2) \]

where since both temperature gradients are negative, the condition in terms of absolute value requires a reversal of the inequality. The last equality introduces a common notation, \( \nabla \equiv d\ln T/d\ln P \), that casts the temperature gradient in terms of the fractional (dimensionless) change relative to the pressure; since pressure is also declining upward, \( \nabla \) is always positive, circumventing the need for taking an absolute value. A key upshot here is that convection will ensue whenever the magnitude of the radiative temperature gradient exceeds that of the adiabatic temperature gradient.

Convection is an inherently complex, 3D dynamical process that generally requires elaborate computer simulations to model accurately. A heuristic, semi-analytic model called “mixing length theory” has been extensively developed. Good summaries are given in the Townsend notes “07-convection.pdf” and MacDonald “Notes_Part_5.pdf”.

Near the stellar surface, the lower density and temperature can make convective transport quite inefficient, so that the actual temperature gradient remains near the radiative value, \( \nabla \approx \nabla_{rad} \).

By contrast, in the dense and hot stellar interior, once convection sets in, it is so efficient at transporting energy that it keeps the local temperature gradient very close to the adiabatic value above which it is triggered, implying \( \nabla \approx \nabla_{ad} \).

Throughout most of a star, one can thus just presume that actual temperature gradient is just set by

\[ \nabla \approx \min(\nabla_{ad}, \nabla_{rad}). \quad (18.3) \]

18.2. Fully convective stars – the Hayashi track for proto-stellar contraction

In hot stars with \( T > 10,000 \text{ K} \), Hydrogen remains fully ionized even to the surface; since there then is no recombination zone to increase the opacity and trigger convection, the
energy transport in their stellar envelopes is by radiative diffusion. In moderately cooler stars like the sun (with \( T_\odot \approx 6000 \text{ K} \)), Hydrogen recombination in a zone just somewhat below the surface induces convection, which thus provides the final transport of energy toward the surface; but since the deeper interior remains ionized and thus non-convective, the general scaling laws derived assuming radiative transport still roughly apply for such solar-type stars.

However, in much cooler stars, with surface temperatures \( T \approx 3500 - 4000 \text{ K} \), the Hydrogen recombination extends deep enough that it triggers other effects that increase the opacity enough to make the entire stellar envelope *fully convective*. Because convection is so much more efficient than radiative diffusion, it can readily bring to the surface any energy generated in the interior – whether produced by gravitational contraction of the envelope, or by nuclear fusion in the core. As such, fully convective stars can have luminosities that greatly exceed the value implied by the \( L \sim M^3 \) scaling law derived in §16.2 (see eqn. 16.3) for stars with radiative envelopes. As discussed in §20, this is a key factor in the high luminosity of cool giant stars that form in the post-main-sequence phases after the exhaustion of Hydrogen fuel in the core.

But it also helps explain the high luminosity of the very cool, early stage of pre-main-sequence evolution, when gravitational contraction of a large *proto-stellar cloud* is providing the energy to make the cloud shine as a *proto-star*. Once the internal pressure generated is sufficient to establish hydro-static equilibrium, its interior becomes fully convective, forcing the proto-star to have this characteristic surface temperature around \( T \approx 3500 - 4000 \text{ K} \).

At early stages the proto-star’s radius is very large, meaning it has a very large luminosity \( L = \sigma_{sb} T^4 4\pi R^2 \). As it contracts, it stays at this temperature, but the declining radius means a declining luminosity. As illustrated in figure 17.1, during this early phase of gravitational contraction, the proto-star thus evolves down a near vertical line in the H-R diagram, dubbed the “Hayashi” track, after the Japanese scientist who first discovered its significance.

Once the radius reaches a level at which the luminosity is near the value predicted by the \( L \sim M^3 \) law, the interior switches from convective to radiative, and so the final contraction to the main sequence makes a sharp turn to a horizontal track (sometimes called the “Henyey” track) with nearly constant luminosity but decreasing surface temperature. The luminosity of this track is set by the stellar mass, according to the \( L \sim M^3 \) law derived for stars with interior energy transport by radiative diffusion. The contraction is halted when the core reaches a temperature for H-fusion, which then stably supplies the luminosity for the main-sequence lifetime.

As detailed in §20, once the star runs out of Hydrogen fuel in its core, its *post-main-
sequence evolution effectively traces backwards along nearly the same track followed during this pre-main-sequence, ultimately leading to the cool, red giant stars seen in the upper right of the H-R diagram, which also have fully convective stellar envelopes.

19. Main Sequence Stars with Core Hydrogen Fusion

The timescale analyses in part I (§10) show that nuclear fusion of Hydrogen into Helium provides a long-lasting energy source that we can associate with main sequence stars in the H-R diagram (§6.3). But what are the requirements for such fusion to occur in the stellar core? And how is this to be related to the luminosity vs. surface temperature scaling for main sequence stars in the HR diagram? In particular, how might this determine the relation between mass and radius? Finally, what does it imply about the lower mass limit for stars to undergo Hydrogen fusion?

Fig. 19.1.— The dominant reaction channel for the proton-proton chain that characterizes hydrogen fusion in the sun and other low-mass stars.
19.1. Core temperature for H-fusion

In stars of a few solar masses or less, Hydrogen fusion occurs through direct proton-proton collision, known thus as p-p burning. Figure 19.1 illustrates the most important of the detailed reaction channels, but the overall result is simply

\[ 4_1^1 H^+ \rightarrow 4He^{+2} + \nu + 2e^+ + E_\gamma, \]  

(19.1)

where \( \nu \) represents a weakly interacting neutrino, which simply escapes the star. The \( 2e^+ \) represents two positively charged “anti-electrons”, or positrons, which quickly annihilate with ordinary electrons, releasing \( \approx 2 \times 2 \times \frac{1}{2} \approx 2 \text{MeV} \) of energy. The rest of the net \( \approx 4 \times 7\text{MeV} \) in energy, representing the mass-energy difference between \( 4H \) vs. one \( He \), is released as \( \gamma \)-rays of energy \( E_\gamma \).

The essential requirement for such p-p fusion is that the thermal kinetic energy \( kT \) of the protons overcome the mutual repulsion of their positive charge \(+e\), to bring the protons to a close separation at which the strong nuclear (attractive) force is able take over, and bind the protons together. For a given temperature \( T \), the minimum separation \( b \) for two protons colliding head-on comes from setting this thermal kinetic energy equal to the electrostatic repulsion energy,

\[ kT = \frac{e^2}{b}. \]  

(19.2)

In particular, if we were to require that this minimum separation be equal to the size of a Helium nucleus, i.e. \( b \approx 1 \text{fm} = 10^{-15} \text{m} \), then from eqn. (19.2) we find that the required temperature is quite extreme, \( T \approx 1.7 \times 10^{10} \text{K} \! \! \! \!.\)

Comparison with the virial scaling (3.13) shows this is more than a thousand times the characteristic virial temperature for the solar interior, \( T_{\text{int}} \approx 13 \text{MK} \). As such, the closest approach between protons in the interior core of the sun is actually more than a thousand times the size of the Helium nucleus, which is thus well outside the scale for operation of the strong nuclear force that keeps the nucleus bound.

The reason that nuclear fusion can nonetheless proceed at such a relatively modest temperature stems again from the uncertainty principle of modern quantum physics. Namely, a proton with thermal energy \( m_p v_{th}^2 / 2 = kT \) has an associated momentum \( p = m_p v_{th} = \sqrt{2m_p kT} \). Within quantum mechanics, it thus has an associated ‘fuzziness’ in position, characterized by its De Broglie wavelength \( \lambda = h/p \), where \( h \) is Planck’s constant. If \( \lambda > b \),

\[ ^{19} \text{In higher mass stars higher core temperatures, H-fusion is catalyzed by nuclear reactions of Hydrogen with Carbon, Oxygen, and Nitrogen, through what is known as the CNO cycle.} \]
then there is a good probability that this waviness of protons will allow them to ‘tunnel’ through the electrostatic repulsion barrier between them, and so find themselves within a nuclear distance at which the strong attractive nuclear force can bind them. Setting $b = \lambda = \hbar/(m_p v_{\text{th}})$ in eqn. (19.2), we can thus obtain an explicit expression for the proton thermal speed needed for nuclear fusion of Hydrogen,

$$v_{\text{th, nuc}} = \frac{2e^2}{\hbar} = 690 \text{ km/s}.$$  \hspace{1cm} (19.3)

Two remarkable aspects of eqn. (19.3) are: (1) this thermal speed for H-fusion depends only on the fundamental physics constants $e$ and $\hbar$, and (2) its numerical value is very nearly equal to the surface escape speed from the sun, $v_{\text{esc}} = \sqrt{2GM_{\odot}/R_{\odot}} = 618 \text{ km/s}$. Recalling the virial scaling (3.13) that says the thermal energy in the stellar interior is comparable to the gravitational binding energy, this means that given the solar mass $M_{\odot}$ the sun has adjusted to just the radius needed for the gravitational binding to give an interior temperature that is hot enough for Hydrogen fusion. For mean molecular weight $\bar{\mu} \approx 0.6m_p$, the mean thermal speed (19.3) implies a core temperature

$$T_{\text{nuc}} = \frac{\mu v_{\text{th, nuc}}^2}{2k} = \frac{1.2m_p e^4}{kh^2} \approx 17 \text{ MK},$$  \hspace{1cm} (19.4)

which now is quite comparable to the interior temperature $T_{\text{int, vir}} \approx 13 \text{ MK}$ obtained by applying the virial scaling (3.13) to the sun.

19.2. Main sequence scalings for radius-mass and luminosity-temperature

If we were to naively apply these same scalings to stars with different masses, then it would suggest all stars along the main sequence should have the same, solar ratio of mass to radius, and thus that the radius should increase linearly with mass, $R \sim M$. In practice, the relation is somewhat sublinear, $R \sim M^{0.7}$. This can be understood by considering that the much higher luminosity of more massive stars, scaling as $L \sim M^3$, means that the core – within which the total fuel available scales just linearly with stellar mass $M$ – must have more vigorous nuclear burning\textsuperscript{20}. The higher core temperature to drive such more vigorous

\textsuperscript{20} Indeed, as already noted, in massive stars the standard, direct proton-proton fusion is augmented by a process called the CNO cycle, in which CNO elements act as a catalyst for H-fusion. Attaching protons to such more highly charged CNO nuclei requires a higher temperature to overcome the stronger electrical repulsion, and this indeed obtains in such massive stars.
H-fusion then requires by the Virial theorem that the mass to radius ratio of such stars must be somewhat higher than for lower mass stars like the sun.

Combining such a sub-linear radius scaling $R \sim M^{0.7}$ with the mass-luminosity scaling $L \sim M^3$ (eqn. 16.3) and the Stefan-Boltzmann relation $L \sim T^4 R^2$, we infer that luminosity should be a quite steep function of surface temperature along the main sequence, viz. $L \sim T^8$. While observed HR diagrams (like that plotted for nearby stars in part I) show the main sequence to have some complex curvature structure, a straight line with $\log L \sim 8 \log T$ does give a rough overall fit, thus providing general support for these simple scaling arguments.

### 19.3. Lower mass limit for hydrogen fusion: Brown Dwarf stars

These nuclear burning scalings can also be used to estimate a minimum stellar mass for Hydrogen fusion. Stars with mass below this minimum are known as Brown Dwarfs. A key new feature of these stars is that their cores become “electron degenerate”, and so no longer follow the simple virial scalings derived above for stars in which the pressure is set by the ideal gas law. Electron degeneracy occurs when the electron number density $n_e$ becomes comparable to cube of the electron De Broglie wavenumber $k_e$,

$$n_e \approx k_e^3 = \frac{1}{\tilde{\lambda}_e^3}, \quad (19.5)$$

with the electron thermal De Broglie (reduced) wavelength,

$$\tilde{\lambda}_e = \frac{h}{p_e} = \frac{\hbar}{\sqrt{2m_e kT}}, \quad (19.6)$$

where $\hbar \equiv h/2\pi$, and the latter equality casts the electron thermal momentum $p_e$ in terms the temperature $T$ and electron mass $m_e$. Assuming a constant density $\rho = M/(4\pi R^3/3) \approx m_p n_e$, we can combine (19.5) and (19.6) with the nuclear temperature (19.4) and the Virial relation (3.13) to obtain a relation for the stellar mass at which a nuclear burning core should become electron degenerate,

$$M_{\text{min, nuc}} = \frac{\sqrt{3/2}}{4\pi^2} \left(\frac{m_p}{m_e}\right)^{3/4} \frac{e^3}{G^{3/2} m_p^2} \approx 0.1 M_\odot. \quad (19.7)$$

Stars with a mass below this minimum should not be able to ignite H-fusion, because electron degeneracy prevents their cores from contracting to a small enough size to reach the $\sim 15$ MK temperature (see eqn. 19.4) required for fusion. In practice, more elaborate computations indicate such Brown Dwarf stars have a limiting mass $M_{BD} \lesssim 0.08 M_\odot$, just slightly below the simple estimate given in (19.7).
Note that, although this minimum mass for H-fusion is limited by electron degeneracy, the actual value is independent of Planck’s constant $h$! In effect, the role of $h$ in the tunneling for H-fusion cancels its role in electron degeneracy. The result is that this Brown-dwarf mass limit depends only on classical parameters like the gravitation constant $G$, the electron charge $e$, and the electron and proton masses $m_e$ and $m_p$.

**Exercise:** Derive eqn. (19.7) from equations cited in the text.

### 19.4. Upper mass limit for stars: the Eddington Limit

Let’s next consider what sets the upper mass limit for observed stars. This is not linked to nuclear burning or degeneracy, but stems from the strong $L \sim M^3$ scaling of luminosity with mass, which, as noted in §16.2, follows from the hydrostatic support and radiative diffusion of the stellar envelope.

In addition to its important general role as a carrier of energy, radiation also has an associated momentum, set by its energy divided by the speed of light $c$. The trapping of radiative energy within a star thus inevitably involves a trapping of its associated momentum, leading to an outward radiative force, or for a given mass, an outward radiative acceleration $g_{\text{rad}}$, that can compete with the star’s gravitational acceleration $g$. For a local radiative energy flux $F$ (energy/time/area), the associated momentum flux (force/area, or pressure) is just $F/c$. The material acceleration resulting from absorbing this radiation depends on the effective cross sectional area $\sigma$ for absorption, divided by the associated material mass $m$, as characterized by the opacity $\kappa$,

$$g_{\text{rad}} = \frac{\sigma F}{mc} = \frac{\kappa F}{c}.$$  \hspace{1cm} (19.8)

For a star of luminosity $L$, the radiative flux at some radial distance $r$ is just $F = L/4\pi r^2$. This gives the radiative acceleration the same inverse-square radial decline as the stellar gravity, $g = GM/r^2$, meaning that it acts as a kind of “anti-gravity”.

Sir Arthur Eddington first noted that, even for a minimal case in which the opacity just comes from free-electron scattering, $\kappa = \kappa_e = 0.2(1+X) \approx 0.34 \, \text{cm}^2\,\text{g}^{-1}$ (with the numerical value for standard (solar) Hydrogen mass fraction $X \approx 0.7$), there is a limiting luminosity, now known as the “Eddington luminosity”, for which the radiative acceleration $g_{\text{rad}} = g$ would completely cancel the stellar gravity,

$$L_{\text{Edd}} = \frac{4\pi GMc}{\kappa_e} = 3.8 \times 10^4 L_\odot \frac{M}{M_\odot}.$$  \hspace{1cm} (19.9)
Any star with \( L > L_{Edd} \) is said to exceed the “Eddington limit”, since even the radiative acceleration from just scattering by free electrons would impart a force that exceeds the stellar gravity, thus implying that the star would no longer be gravitationally bound!

For main sequence stars that follow the \( L \sim M^3 \) scaling, setting \( L = L_{Edd} \) yields an estimate for an upper mass limit at which the star would reach this Eddington limit,

\[
M_{\text{max},Edd} \approx 195 M_\odot,
\]

where \( 195 \approx \sqrt{3.8 \times 10^7} \). This agrees quite well with modern empirical estimates for the most massive observed stars, which are in the range 150-300 \( M_\odot \).

Actually, as stars approach this Eddington limit, the radiation pressure alters the hydrostatic structure of the envelope, causing the mass-luminosity relation to weaken toward a linear scaling, \( L \sim M \), and so allowing in principle for stars with even with mass \( M > M_{\text{max},Edd} \) to remain bound. In practice, such stars are subject to “photon bubble” instabilities, much as occurs whenever a heavy fluid (in this case the stellar gas) is supported by a lighter one (here the radiation). Very massive stars near this Eddington limit thus tend to be highly variable, often with episodes of large ejection of mass that effectively keeps the stellar mass near or below the \( M_{\text{max},Edd} \approx 195 M_\odot \) limit.

20. Post-Main-Sequence Evolution

As a star ages, more and more of the Hydrogen in its core becomes consumed by fusion into Helium. Once this core Hydrogen is exhausted, how does the star react and adjust? Without the H-fusion to supply its luminosity, one might think that perhaps the star would simply shrink, cool and dim, and so die out, much as a candle when all its wax is used up.

Instead it turns out that stars at this post-main-sequence stage of life actually start to expand, at first keeping roughly the same luminosity and so becoming cooler at the surface, but eventually becoming much brighter giant or supergiant stars, shining with a luminosity that can be thousands or even tens of thousands that of their core-H-burning main sequence.

Figure 20.1 illustrates the post-MS evolution for the sun (left) and stars with mass up to \( 10 M_\odot \) (right). Let us focus our initial attention on solar type stars with \( M \lesssim 8 M_\odot \).
20.1. Post-MS evolution and death of solar-type stars with $M \lesssim 8M_\odot$

20.1.1. Core-Hydrogen burning and evolution to the Red Giant branch

The apparently counterintuitive post-main-sequence adjustment of stars can actually be understood through the same basic principals used to understand their initial, pre-main-sequence evolution. When the core runs out of Hydrogen fuel, the lack of energy generation does indeed cause the core itself to contract. But the result is to make this core even denser and hotter. Then, much as the hot coals at the heart of wood fire help burn the wood fuel around it much faster, the higher temperature of a contracted stellar core actually makes the overlying shell of Hydrogen fuel around the core burn even more vigorously!

Now, unlike during the main sequence – when there is an essential regulation or compatibility between the luminosity generated in the core and the luminosity that the radiative envelope is able to transport to the stellar surface –, this shell-burning core is actually over-luminous relative to the envelope luminosity that is set by the $L \sim M^3$ scaling law. As such, instead of emitting this core luminosity as surface radiation, the excess energy acts to re-inflate the star, in effect doing work against gravity to reverse the Kelvin-Helmholtz contraction that occurred during the star’s pre-main-sequence evolution. Initially, the radiative envelope keeps the luminosity fixed so that, as the star expands, the surface temperature again declines, with the star thus again evolving horizontally on the H-R diagram, this time...
from left to right.

But as the surface temperature approaches the limiting value $T \approx 3500 - 4000$ K, the envelope again becomes more and more convective, which thus now allows this full high-luminosity of the H-shell-burning core to be transported to the surface. The star's luminosity thus increases, with now the temperature staying nearly constant at the cool value for the Hayashi limit. In the H-R diagram, the star essentially climbs back up the Hayashi track, eventually reaching the region of the cool, red giants in the upper right of the H-R diagram.

The above describes a general process for all stars, but the specifics depend on the stellar mass. For masses less than the sun, the main sequence temperature is already quite close to the cool limit, so evolution can proceed almost directly vertically up the Hayashi track. For masses much greater than the sun, the luminosity and temperature on the main sequence are both much higher, and so the horizontal evolutionary phase is more sustained. And since the luminosity is already very high, these stars become red supergiants without ever having to reach or climb the Hayashi track.

20.1.2. Helium flash and core-Helium burning on the Horizontal Branch

This Hydrogen-shell burning also has the effect of increasing further the temperature of the stellar core, and eventually this reaches a level where the fusion of the Helium itself becomes possible, through what’s known as the “triple-$\alpha$ process”\textsuperscript{21},

$$3 \, ^4\text{He}^+ + ^8\text{Be}^+ \rightarrow ^{12}\text{C}^+ + 4 \, ^4\text{He}^+.$$ \hspace{1cm} (20.1)

The direct fusion of two $^4\text{He}^+$ nuclei initially make an unstable nucleus of Berylium ($^8\text{Be}^+$), which usually just decays back into Helium. But if the density and temperature are sufficiently high, then during the brief lifetime of the unstable Berylium nucleus, another Helium can fuse with it to make a very stable Carbon nucleus $^{12}\text{C}^+$. Since the final step of fusing $^4\text{He}^+$ and $^8\text{Be}^+$ involves overcoming an electrostatic repulsion that is $2 \times 4 = 8$ times higher than for proton-proton (p-p) fusion of Hydrogen, He-fusion requires a much higher core temperature, $T_{\text{He}} \approx 120$ MK.

In stars with more than a few solar masses, this ignition of the Helium in the core occurs gradually, since the higher core temperature from the addition of He-burning increases the gas pressure, making the core tend to expand in a way that regulates the burning rate.

\textsuperscript{21}since Helium nuclei are sometimes referred as “$\alpha$-particles”
In contrast, for the sun and other stars with masses $M < 2M_\odot$, the number density of electrons $n_e$ in the helium core is so high\(^{22}\) that their core becomes *electron degenerate*. As discussed in \S 19.3 for the Brown dwarf stars that define the lower mass limit for H-burning, electron degeneracy occurs when the mean distance between electrons $\sim n_e^{-1/3}$ becomes comparable to the DeBroglie wavelength $\bar{\lambda}_e = \hbar/p_e$. The properties of such degeneracy are discussed further in \S 20.1.4 on the degenerate white-dwarf end states of solar-type stars. But in the present context a key point is that, unlike for the ideal gas law, the pressure of a degenerate gas becomes *independent of temperature*! This means that any heating of the core no longer causes a pressure-driven expansion that normally regulates the nuclear burning of a core governed by the ideal gas law.

Without then the regulation from such thermal pressure-driven expansion, the ignition of He burning leads to a *Helium flash*, in which the entire degenerate core of Helium is fused into Carbon over a very short timescale. This flash marks the “tip” of the Red Giant Branch (RGB) in the H-R diagram, but somewhat surprisingly, the sudden addition of energy is largely absorbed by the expansion of the core and the immediate overlying stellar envelope. The star thus quickly settles down to a more quiescent phase of He-burning. Because the He burning causes the core to expand, the shell burning of H actually declines, causing the luminosity to decrease from the tip of the Red Giant branch, where the He flash occurs, to a somewhat hotter, dimmer region known as the “Horizontal Branch” in the H-R diagram.

This Horizontal Branch (HB) can be loosely thought of as the He-burning analog of the H-burning Main Sequence (MS), but a key difference is that it lasts a much shorter time, typically only 10 to 100 *million years*, much less than the many *billion years* for a solar mass star on the MS. This is partly because the luminosity for HB stars is so much higher than for a similar mass on the MS, implying a much higher burn rate of fuel. But another factor is that the energy yield per-unit-mass, $\epsilon$, for He-fusion to Carbon is about a tenth of that for H-fusion to Helium, viz. about $\epsilon_{He} \approx 0.06\%$ vs. the $\epsilon_{H} \approx 0.7\%$ for H-burning. With lower energy produced, and higher rate of energy lost in luminosity, the lifetime is accordingly shorter.

---

\(^{22}\)Recall that on the main sequence the radii of stars is (very) scales roughly as a mildly sub-linear power of the mass, i.e. $R \sim M^p$, with $p \approx 2/3$. But since density scales as $\rho \sim M/R^3$, the *density* of low mass stars tends generally to be higher than in high-mass stars, roughly scaling as $\rho \sim M^{1-3p} \sim 1/M$. This overall scaling of average stellar density also applies to the relative densities of stellar cores, and so represents one reason why the cores of low-mass stars tend to become electron degenerate, while those of higher mass stars do not.
20.1.3. Asymptotic Giant Branch to Planetary Nebula to White Dwarf

Once the core runs out of Helium, He-burning also shifts to an inner shell around the core, which itself is still surrounded by a outer shell of more vigorous H-burning. This again tends to increase the core luminosity, but now since the star is cool and thus mostly convective, this energy is mostly transported to the surface with only a modest further expansion of the stellar radius. This causes the star to again climb in luminosity along what’s called the “Asymptotic Giant Branch” (AGB), which parallels the Hayashi track at just a somewhat hotter surface temperature.

In the sun and stars of somewhat higher mass, up to $M \lesssim 8M_{\odot}$, there can be further ignition of the Carbon to fuse with Helium to form Oxygen. But further synthesis up the periodic table requires overcoming the greater electrical repulsion of more highly charge nuclei. This in turn requires a temperature higher than occurs in the cores lower mass stars, for which the onset of electron degeneracy prevents contraction to a denser, hotter core. Further core burning thus ceases, leaving the core as an inert, degenerate ball of C and O, with final mass on order of $1M_{\odot}$, with the remaining mass contained in the surrounding envelope of mostly Hydrogen.

But such AGB stars tend also to be pulsationally unstable, and because of the very low surface gravity, such pulsations can over time actually eject the entire stellar envelope. This forms a circumstellar nebula that is heated and ionized by the very hot remnant core. As seen in the left panel of figure 20.5, the resulting circular nebular emission glow somewhat resembles the visible disk of a planet, so these are called “planetary nebulae”, though they really have nothing much to do with actual planets. After a few thousand years, the planetary nebula dissipates, leaving behind just the degenerate remnant core, a white dwarf star.

20.1.4. White Dwarf stars

The degenerate nature of white dwarf stars endows them with some rather peculiar, even extreme properties. As noted, they typically consist of roughly a solar mass of C and O, but have a radius comparable to that of the earth, $R_e \approx R_{\odot}/100$. This small radius makes them very dense, with $\rho_{wd} \approx 10^6\rho_{\odot} \approx 10^6$ g/cm$^3$, i.e. about a million times (!) the density of water, or the density of normal main-sequence stars like the sun. It also gives them very strong surface gravity, with $g_{wd} \approx 10^4g_{\odot} \approx 10^6$ m/s$^2$, or about 100,000 times earth’s gravity!

As noted in §19.3 for the Brown dwarf stars that define the lower mass limit for H-burning, gas becomes electron degenerate when the electron number density $n_e$ becomes so high that the mean distance between electrons becomes comparable to their reduced De
Broglie wavelength,
\[ n_e^{-1/3} \approx \bar{\lambda} \equiv \frac{\hbar}{P_e} = \frac{\hbar}{m_e v_e}, \]  
(20.2)
where the electron thermal momentum \( p_e \) equals the product of its mass \( m_e \) and thermal speed \( v_e \), and \( \hbar \equiv \hbar/2\pi \) is the reduced Planck constant. The associated electron pressure is
\[ P_e = n_e v_e p_e = n_e^{5/3} \frac{\hbar^2}{m_e} = \rho Z \left( \frac{Am_p}{Z} \right)^{5/3} \frac{\hbar^2}{m_e}, \]  
(20.3)
where the last equality uses the relation between electron density and mass density, \( \rho = n_e Am_p/Z \), with \( Z \) and \( Am_p \) the average nuclear charge and atomic mass. For example, for a Carbon white dwarf, the atomic number \( Z = 6 \) gives the number of free (ionized) electrons needed to balance the \(+Z = +6\) charge of the Carbon nucleus, while the atomic weight \( Am_p = 12m_p \) gives the associated mass from the C atoms.

The hydrostatic equilibrium (cf. eqn. 3.3) for pressure gradient support against gravity then requires for a white dwarf star with mass \( M_{wd} \) and radius \( R_{wd} \),
\[ \frac{P_e}{R_{wd}} \approx \rho \frac{G M_{wd}}{R_{wd}^2}. \]  
(20.4)
Using the density scaling \( \rho \sim M_{wd}/R_{wd}^3 \), we can combine (20.3) and (20.4) to solve for a relation between the white-dwarf radius and its mass,
\[ R_{wd} = \frac{1}{GM_{wd}^{1/3}} \frac{h^2}{m_e} \left( \frac{Z}{Am_p} \right)^{5/3} \approx 0.01 R_\odot \left( \frac{M_\odot}{M_{wd}} \right)^{1/3}, \]  
(20.5)
where the approximate evaluation uses the fact that for both C and O the ratio \( Z/A = 1/2 \). For a typical mass of order the solar mass, we see that a white dwarf is very compact, comparable to the radius of the earth, \( R_e \approx 0.01 R_\odot \). But note that this radius actually decreases with increasing mass.

20.1.5. Chandasekhar limit for white-dwarf mass: \( M < 1.4M_\odot \)

This fact that white-dwarf radii decrease with higher mass means that, to provide the higher pressure to support the stronger gravity, the electron speed \( v_e \) must strongly increase with mass. Indeed, at some point this speed approaches the speed of light, \( v_e \approx c \), implying that the associated electron pressure now takes the scaling (cf. eqn. 20.3),
\[ P_e = n_e c p_e = n_e^{4/3} \hbar c = \left( \frac{\rho Z}{Am_p} \right)^{4/3} \hbar c. \]  
(20.6)
Applying this in the hydrostatic relation (20.4), we now find that the radius \( R \) cancels, and we instead can solve for a upper limit for a white dwarf’s mass,

\[
M_{wd} \leq M_{ch} = \left( \frac{\hbar c}{G} \right)^{3/2} \left( \frac{Z}{A m_p} \right)^2 \approx 1.4 M_\odot.
\]  (20.7)

where the subscript refers to “Chandrasekhar”, the astrophysicist who first derived this mass limit.

As discussed in later sections, when accretion of matter from a binary companion puts a white dwarf over this limit, it triggers an enormous “white-dwarf supernova” explosion, with a large, well-defined peak luminosity, \( L \approx 10^{10} L_\odot \). This provides a very bright standard candle that can be used to determine distances as far as a Gpc, giving a key way to calibrate the expansion rate of the universe.

But in the present context, this limit means that sufficiently massive stars with cores above this mass cannot end their lives as a white dwarf. Instead, they end as violent “core-collapse supernovae”, leaving behind an even more compact final remnant, either a neutron star or black hole, as we discuss next.

### 20.2. Post-MS evolution and death of high-mass stars with \( M > 8M_\odot \)

#### 20.2.1. Multiple shell burning and horizontal loops in H-R diagram

The post-main-sequence evolution of stars with higher initial mass, \( M > 8M_\odot \) has some distinct differences from that outlined above for solar and intermediate mass stars. Upon exhaustion of H-fuel at the end of the main sequence, such stars again expand in radius because of the over-luminosity of H-shell burning. But the high luminosity and high surface temperature on the main sequence means that their stellar envelopes remain radiative even as they expand, never reaching the cool temperatures that force a climb up the Hayashi track. Instead, their evolution tends to keep near the constant luminosity set by \( L \sim M^3 \) scaling for the star’s mass, so evolving horizontally to the right on the H-R diagram.

For a combination of reasons (lower mean stellar density, stronger gravitational confinement leading to higher core pressures and temperatures), the core density of high-mass stars never becomes high enough to become electron degenerate. As the cores contract with each stage of burning, they thus reach an ever-higher core temperature. This now makes it possible to overcome the increasingly strong electrical repulsion of more highly charged, higher elements, allowing nucleosynthesis to proceed up the period table all the way to Iron, which is the most stable nucleus.
Fig. 20.2.— Binding energy per nucleon plotted vs. the number of nucleons in a nucleus. The fusion of light elements moves nuclei to the right, releasing the energy of nuclear burning in the very hot dense cores of stars, but only up to formation of the most stable nucleus, for iron (Fe), with atomic number $A = 56$. In the sudden core collapse of massive-star supernovae, the copious energy available synthesizes elements even beyond iron. The fission of such heavy elements leads to lower-mass nuclei toward the left. The energy released is what powers nuclear fission reactors here on earth.

However, each such higher level of nucleosynthesis yields proportionally less and less energy. This can be seen from the plot in figure 20.2 of the binding energy per nucleon vs. the number of nucleons in a nucleus. The jump from H to He yields 7.1 MeV per nucleon, which relative to a nucleon mass of 931 MeV represents a percentage energy release of about 0.7%, as noted above. But from He to C the release is just $7.7 - 7.1 = 0.6$ MeV, representing an energy efficiency of just 0.06%. As the curve flattens out, the fractional energy release become even less, until for elements beyond Iron, further fusion would require the addition of energy.

For such massive stars, the final stages of post-main sequence evolution are thus characterized by an increasingly massive Iron core that can no longer produce any energy by
further fusion. But fusion still occurs in a surrounding series of shells, somewhat like an onion skin, with higher elements fusing in the innermost, hottest shells, and outer shells fusing lower elements, extending to an outermost shell of H-burning (see figure 20.3).

Fig. 20.3.— The “onion-skin” layering of the core of a 25-$M_\odot$ star just before supernovae core collapse, illustrating the various stages of nuclear burning in shells around the inert iron core. The right box shows the decreasing duration for each higher stage of burning.

20.2.2. Core-collapse supernovae

With the build-up of Iron in the core, there is an increasingly strong gravity, but without the further fusion-generated energy to keep the temperature high, the core pressure becomes unable to support the mass above. This eventually leads to a catastrophic core collapse, halted only when the electrons merge with the protons in the Iron nuclei to make the entire core into a collection of neutrons, with a density so high that they now actually become neutron degenerate. The “stiffness” of this neutron-degenerate core leads to a “rebound” in the collapse, with gravitational release from the core contraction now powering an explosion that blows away the entire outer regions of the star, with the stellar ejecta reaching speeds of about 10% the speed of light! This ejecta contains Iron and other heavy elements, including even those beyond Iron that are fused in less than a second of the explosion by the enormous energy and temperatures. While elements up to Oxygen can also be synthesized in low-mass stars, all the elements heavier than Iron – e.g. Nickel, Lead, all the way up the periodic table to heaviest naturally occurring element, Uranium – originated in massive-star, core-collapse
supernova explosions. For a few weeks, the luminosity of such a supernova can equal or exceed that of a whole galaxy, up to $\sim 10^{12}L_\odot$!

Though the dividing line is not exact, it is thought that all stars with initial masses $M > 8M_\odot$ will end their lives with such a core-collapse supernova, instead of following the track, AGB $\rightarrow$ PN $\rightarrow$ White Dwarf, for stars with initial mass $M < 8M_\odot$. Stars with initial masses $8M_\odot < M \lesssim 30M_\odot$ are thought to leave behind a neutron star remnant, as discussed next. But as discussed in §20.2.4 below, such neutron star remnants have their own upper mass limit of $M_{ns} \lesssim 3M_\odot$, beyond which the gravity becomes so strong that not even the degenerate pressure from neutrons can prevent a further collapse, this time forming a black hole. This is thought to be the final core remnant for the most massive stars, those with initial mass $M \gtrsim 30M_\odot$.

20.2.3. Neutron stars

Neutron stars are even more bizarrely extreme than white dwarfs. With a mass typically about twice a solar mass, they have a radius comparable to a small city, $R_{ns} \approx 10\,\text{km}$, about a factor 600 smaller than even a white dwarf, implying a density that is about $10^8$ higher, and a surface gravity more than $10^5$ times higher!

The radius and mass properties associated with neutron degeneracy can be derived by a procedure completely analogous to that used in §§20.1.4 and 20.1.5 for white dwarfs, just substituting now the electron mass with the neutron mass, $m_e \rightarrow m_n \approx m_p$, and setting $Z/A = 1$. The radius-mass relation thus now becomes (cf. eqn. 20.5),

$$R_{ns} = \frac{1}{G M_{ns}^{1/3}} \frac{\hbar^2}{m_n^{8/3}} \approx \left[ \frac{10\,\text{km}}{\frac{M_\odot}{M_{ns}}} \right]^{1/3}. \quad (20.8)$$

Note again that, as in the case of an electron-degenerate white dwarf, this neutron-star radius also decreases with increasing mass.

For the same reasons that lead to the upper mass limit for white dwarfs, for sufficiently high mass the neutrons become relativistic, leading now to an upper mass limit for neutron stars (cf. eqn. 20.7),

$$M_{ns} \leq M_{lim} = \left( \frac{\hbar c}{G} \right)^{3/2} \left( \frac{1}{m_p} \right)^2 \approx \left[ 3M_\odot \right], \quad (20.9)$$

which is just a factor $\sim A/Z = 2$ larger than the Chandrasekhar mass for white dwarfs. Neutron stars above this mass will again collapse, this time forming a black hole.
20.2.4. Black Holes

Black holes are objects for which the gravity is so strong that not even light itself can escape. A proper treatment requires General Relativity, Einstein’s radical theory of gravity that supplants Newton’s theory of universal gravitation, and extends it to the limit of very strong gravity. But we can nonetheless use Newton’s theory to derive some basic scalings. In particular, for a given mass $M$, a characteristic radius for which the Newtonian escape speed is equal to speed of light, $v_{\text{esc}} = c$, is just

$$R_{\text{bh}} = \frac{2GM}{c^2} \approx 3 \text{ km } M_{\odot},$$

which is commonly known as the “Schwarzschild radius”, but within astrophysical work is often referred to as the “gravitational radius”, sometimes dubbed $R_g$.

Since the speed of light is the highest speed possible, any object within this Schwarzschild radius of a given mass $M$ can never escape the gravitational binding with that mass. In terms of Einstein’s General Theory of Relativity, mass acts to bend space and time, much the way a bowling ball bends the surface of a trampoline. And much as a sufficiently dense, heavy ball could rip a hole in the trampoline, for objects with mass concentrated within a radius $R_{\text{bh}}$, the bending becomes so extreme that it effectively punctures a hole in space-time. Since not even light can ever escape from this hole, it is completely black, absorbing any light or matter that falls in, but never emitting any light from the hole itself. This the origin of the term “black hole”.

Stellar-mass black holes with $M \gtrsim 3M_{\odot}$ form from the deaths of massive stars. If left over from a single star, they are hard or even impossible to detect, since by definition they don’t emit light.

However, in a binary system, the presence of a black hole can be indirectly inferred by observing the orbital motion (visually or spectroscopically via the Doppler effect) of the luminous companion star.

Moreover, when that companion star becomes a giant, it can, if it is close enough, transfer mass onto the black hole. Rather than falling directly into the hole, the conservation of the angular momentum from the stellar orbit requires that the matter first feed an orbiting accretion disk. By the Virial theorem, half the gravitational energy goes into kinetic energy of orbit, but the other half is dissipated to heat the disk, which by the blackbody law then emits it as radiation.

The luminosity of such black-hole accretion disks can be very large. For a black hole of mass $M_{\text{bh}}$ accreting mass at a rate $\dot{M}_a$ to a radius $R_a$ that is near the Schwarzschild radius
$R_{bh}$, the luminosity generated is

$$L_{\text{disk}} = \frac{GM_{bh}\dot{M}_a}{2R_a} = \frac{R_{bh}}{4R_a} \dot{M}_a c^2 \equiv \epsilon \dot{M}_a c^2.$$  \hfill (20.11)

The latter two equalities define the efficiency $\epsilon \equiv R_{bh}/4R_a$ for converting the rest-mass-energy of the accreted matter into luminosity. For accretion radii approaching the Schwarzschild radius, $R_a \approx R_{bh}$, this efficiency can be as high as $\epsilon \approx 0.25$, implying 25% of the accreted matter-energy is converted into radiation. By comparison, for H-fusion of a MS star the overall conversion efficiency is about 0.07%, representing the $\sim 10\%$ core mass that is sufficiently hot for H-fusion at a specific efficiency, $\epsilon_H = 0.007 = 0.7\%$.

![Fig. 20.4.— Artist depiction of mass transfer onto accretion disk around black hole in Cygnus X-1.](image)

**Quick Question 18-1:**

(a) Because of general relativistic effects, it turns out the lowest stable orbit around a black hole is at radius of $3R_{bh}$. What is the luminosity efficiency for accreting to this radius?

(b) What is the accretion luminosity, in $L_\odot$, for a mass accretion rate $\dot{M}_a = 10^{-6}M_\odot$/yr to this radius?

(c) **Challenge problem**: For a black hole with mass $M_{bh} = 3M_\odot$, use the Stefan-Bolzmann law to derive the radiative flux that would balance the local
gravitational heating at this radius \( r = 3R_{bh} \), and then solve for the local black-body temperature \( T(r = 3R_{bh}) \). Express this first as a ratio to sun’s surface temperature \( T_\odot \), and then also in Kelvin.

**Quick Question 18-2:** Use Wien’s law to compute the peak wavelength (in nm) of thermal emission from the inner region of an accretion disk with temperature \( T = 10^7 \text{ K} \). What is the energy (in eV) of a photon with this wavelength? Now also answer both questions for \( T = 10^{10} \text{ K} \). What parts of the electromagnetic spectrum do these photon wavelengths/energies correspond to?

In the inner disk, the associated blackbody temperature can reach \( 10^7 \text{ K} \) or more (see challenge problem); and at the very inner disk edge, dissipation of the orbital energy can heat material to even more extreme temperatures, up to \( \sim 10^{10} \text{ K} \). The associated radiation is very high energy, as seen from the Quick-Question calculations. By studying this high-energy radiation, we can infer the presence and basic properties (mass, even rotation rate) of black holes in such binary systems, even though we can’t see the black hole itself.

Figure 20.4 shows an artist depiction of the mass transfer accretion in the high-mass X-ray binary Cygnus X-1, thought to be the clearest example of a stellar-remnant black hole, estimated in this case to have a mass \( M_{bh} > 10M_\odot \) that is well above the \( M_{\text{lim}} \approx 3M_\odot \) upper limit for a neutron star (cf. eqn. 20.9).

20.3. Observations of stellar remnants

It is possible to observe directly all three types of stellar remnants:

1. **Planetary Nebula and White Dwarf stars**

    Stars with initial mass \( M < 8M_\odot \) evolve to an AGB star that ejects the outer stellar envelope to form a Planetary Nebula (PN), with a hot stellar core of mass below the Chandrasek mass, \( M_{wd} < M_{ch} = 1.4M_\odot \). Once the nebula dissipates, this leave behind a White Dwarf (WD). White dwarf stars are very hot, but with such a small radius that their luminosity is very low, placing them on the lower left of the H-R diagram.

    The excitation and ionization of the gas in the surrounding PN makes it shine with an emission line spectrum, with the wavelength-specific emission of various ion species giving it range of vivid colors or hues. Figure 20.5 shows that these PN can thus be visually quite striking, with spherical emission nebula from single stars (left), or very complex geometric forms (right) for stars in binary systems.
Fig. 20.5.— Left: M57, known as the Ring Nebula, provides a vivid example of a spherically symmetric Planetary Nebula. The central hot star is the remnant of the stellar core, and after the nebula dissipates, it will be left as a White Dwarf star. Right: A gallery of planetary nebulae, showing the remarkable variety of shapes that probably stem from interaction of the stellar ejecta with a binary companion, or perhaps even with the original star’s planetary system.

2. Neutron stars and Pulsars

Stars with initial masses in the range \(8M_\odot < M \lesssim 30M_\odot\) end their lives as a core collapse supernova that leaves behind a neutron star with mass \(1.4M_\odot < M_{ns} < 3M_\odot\). The conservation of angular momentum during the collapse to such a small size (\(\sim 10\, \text{km}\)) makes them rotate very rapidly, often many times a second! This also generates a strong magnetic field, and when the polar axis of this field points toward earth, it leads to a strong pulse of beamed radiation in the radio to optical to even X-rays. This is observed as a pulsar.

One of the best known examples is the Crab pulsar, which lies at the center of the Crab Nebula, the remnant from a core-collapse supernova that was observed by Chinese astronomers in 1054 AD. Figure 20.6 shows images of this Crab nebula in the optical region (left) and in a composite of images (right) in the optical (red) and X-ray (blue) wavebands.

3. Black holes and X-ray binary systems

Finally, stars with initial masses \(M \gtrsim 30M_\odot\) end their lives with a core collapse supernova that now leaves behind a black hole with mass \(M_{bh} > 3M_\odot\). As noted, in
single stars, these are difficult or impossible to observe, because they emit no light; but in binary systems, accretion of material from the other star can power a bright accretion disk around the black hole that radiates in high-energy bands like X-rays and even $\gamma$-rays. Figure 20.4 shows an artist depiction of accretion onto a black hole in the high-mass X-ray binary Cygnus X1.

Fig. 20.6.— Left: Optical image of the Crab Nebula, showing the remnant from a core-collapse supernova whose explosion was observed by Chinese astronomers in 1054. Right: A composite zoomed-in image of the central region Crab Nebula, showing the optical (red) image superimposed with an X-ray (blue) image made by NASA’s Chandra X-ray observatory. The bright star at the nebular center is the Crab pulsar, a rapidly rotating neutron star that was left over from the supernova explosion.