

Apparent suppression of turbulent magnetic dynamo action by a dc magnetic field

David C. Montgomery^{a)}

Department of Physics and Astronomy, Dartmouth College, Hanover, New Hampshire 03755-3528

W. H. Matthaeus, L. J. Milano, and P. Dmitruk

Bartol Research Institute, University of Delaware, Newark, Delaware 19716

(Received 5 December 2001; accepted 25 January 2002)

Numerical studies of the effect of a dc magnetic field on dynamo action (development of magnetic fields with large spatial scales), due to helically-driven magnetohydrodynamic turbulence, are reported. The apparent effect of the dc magnetic field is to suppress the dynamo action, above a relatively low threshold. However, the possibility that the suppression results from an improper combination of rectangular triply spatially-periodic boundary conditions and a uniform dc magnetic field is addressed: heretofore a common and convenient computational convention in turbulence investigations. Physical reasons for the observed suppression are suggested. Other geometries and boundary conditions are offered, for which the dynamo action is expected not to be suppressed by the presence of a dc magnetic field component. © 2002 American Institute of Physics.

[DOI: 10.1063/1.1462631]

I. INTRODUCTION

The spontaneous development of large-scale magnetic fields in astrophysical or geophysical settings has been a problem of interest at least since the time of Gauss, and remains imperfectly understood. One promising candidate for a likely basic and universal “dynamo” process has been the turbulent inverse cascade of magnetic helicity,^{1–3} a topic which has received considerable attention among turbulence theorists. In broad terms, a mechanical source of helical turbulent excitations (often combining thermal convection and rotational properties of the system) is conjectured to excite magnetic turbulence at small spatial scales by a variety of mechanisms such as flux tube stretching. The magnetic turbulence then inversely cascades toward ever-larger scales because of certain statistical properties of the magnetohydrodynamic (MHD) equations, properties that are mathematically understandable but are not altogether intuitive. Virtually no experimental verification of the mechanism has been available, and most of the evidence for the process that has accumulated has come largely from computationally-intensive numerical solutions of the MHD equations, most notably those of Meneguzzi *et al.*⁴ The effect is inherently three-dimensional (3D), because of the involvement of magnetic helicity, although an analogous process driven by inverse transfer of mean square magnetic potential (mean square flux function) is observed in two-dimensional (2D) computations and simulations.

The situation considered is related to, but not quantitatively well represented by, the so-called “alpha effect,” which necessarily assumes large gaps in the magnetic energy spectrum across which the magnetic excitations are supposed to jump as a consequence of microscopic nonlinear

processes.^{5–7} Such numerical evidence as has been presented has always shown that any such initial spectral gap quickly fills in and disappears, and the energy transfer into any band in wavenumber space thereafter tends to be from adjacent wavenumber bands, not from remote ones. Nevertheless, the alpha effect should be considered to stand as the precursor or first hint of dynamo action through the inverse cascade of magnetic helicity.

The investigations reported here were intended to explore the effect of an imposed dc magnetic field on the dynamo action associated with the inverse magnetic helicity cascade. Sometimes in astrophysical cases, and particularly in laboratory situations such as the reversed-field pinch, MHD turbulence takes place in the presence of strong dc magnetic fields whose source currents are external to the magnetofluid. Numerical verification of inverse helicity cascades and their attendant dynamo action of the kind reported by Meneguzzi *et al.*⁴ have generally been achieved by computations in three-dimensional rectangular periodic boundary conditions involving no such imposed dc magnetic fields. It is of interest to see how the inverse helicity cascade and the supposed dynamo process might be affected by the presence of externally-imposed dc magnetic fields, particularly since it is known that dc magnetic fields strongly affect MHD turbulence in other contexts, rendering it highly anisotropic.^{8–11} Some recent computations¹² examine effects of an external field on the alpha parameter, but not its effects on inverse cascade. The subject appears to be far from complete, and considerable controversy persists regarding the nature of the dynamo saturation process.^{12–14} For this reason, we have undertaken the apparently straightforward task of repeating the computations of Meneguzzi *et al.*⁴ but in the presence of an imposed spatially uniform dc magnetic field, to see what changes that field would produce.

In Sec. II, we report the results of this computation. At

^{a)}Electronic mail: david.c.montgomery@dartmouth.edu

first sight, they are surprising in that the presence of the dc magnetic field effectively suppresses the dynamo action at fairly low levels, in this conventional framework that has become standard for MHD turbulence theory and computation.

In Sec. III we discuss the reasons why we believe the conclusions are dependent on certain artificialities and inconsistencies imbedded in the standard formulation's geometry in this case. Finally, in Sec. IV, we offer suggestions for future possibilities for turbulence computations in alternative geometrical settings where we believe a more accurate set of conclusions concerning the effects of dc magnetic fields in the dynamo problem may be drawn.

II. APPARENT DYNAMO SUPPRESSION

We employ a fully dealiased spectral code¹⁵ for solving the equations of incompressible MHD with uniform mass density. These are basically an equation of motion, Faraday's law, and an Ohm's law, with velocity fields and magnetic fields which are both solenoidal. Written out in detail, we have

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} + \nu \nabla^2 \mathbf{v} + \mathbf{f}, \quad (1)$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad (2)$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}. \quad (3)$$

The symbols are as follows. \mathbf{B} is the magnetic field (in Alfvénic velocity units), \mathbf{v} is the velocity field, and p is the pressure. \mathbf{E} is the electric field and \mathbf{j} is the electric current density, given by the curl of \mathbf{B} . The dimensionless viscosity and resistivity are ν and η , respectively, and are in effect the reciprocals of Reynolds-like numbers, mechanical and magnetic. For the present computations, ν and η are being chosen equal (unit magnetic Prandtl number). Both \mathbf{B} and \mathbf{v} have zero divergences. \mathbf{B} could be written as the curl of a vector potential \mathbf{A} , if desired. We will not greatly emphasize the role of \mathbf{A} , because of ambiguities in its role in possible definitions of magnetic helicity in the presence of a dc magnetic field.^{16,17} These ambiguities are not central to the points we intend to make, which pertain primarily to the appearance or nonappearance of long-wavelength magnetic field spectral components—directly computable without reference to \mathbf{A} . The inhomogeneous mechanical forcing term \mathbf{f} , on the right hand side of Eq. (1), is a given, solenoidal, random, vector function designed to inject mechanical helicity and is intended to mimic whatever mechanically turbulent processes one may invoke as the source of the velocity-field excitations.

We attempt to work in regimes in which the mean kinetic energy per unit volume and rms velocity field are of order unity, so that our Reynolds-like numbers are based on length scales that are roughly one-sixth of the basic box size. All fields are expanded in three-dimensional Fourier series in a cubical box of edge length 2π , so that rectangular periodic boundary conditions are being assumed in all three directions. The magnetic field will be written as $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, where \mathbf{B}_0 is a uniform constant and the spatial average of the variable magnetic field, \mathbf{b} , is zero.

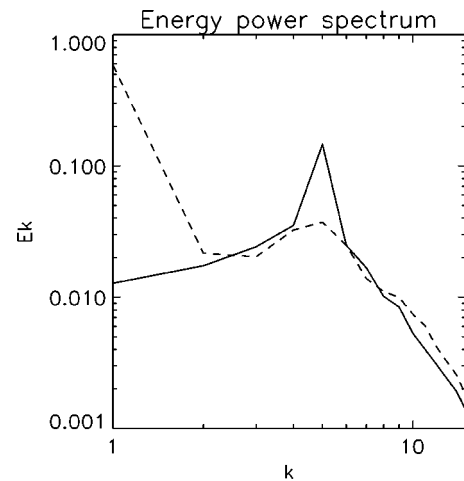


FIG. 1. Evolved ($t=1000$) kinetic energy spectrum (solid line) and magnetic energy spectrum (dashed line) in the presence of no dc magnetic field ($B_0=0$) with helical mechanical forcing. This spectrum is totally dominated by the $k=1$ modes.

It has often been thought desirable to study MHD turbulence processes at as high Reynolds-like numbers as possible (i.e., ν and η as small as possible), but in three dimensions, long-time computations of this type require high spatial resolution and make expensive demands on computer time. We have concluded that the points we wish to make here can be made convincingly at lower Reynolds-like numbers, and the runs to be reported have all been carried out with ν and $\eta = 0.01$, while spatial resolution of only 32^3 has been employed. It would be desirable to repeat these runs with lower ν and η and with greater spatial resolution, but we would not expect the conclusions we will draw here to be changed by doing so.

We will illustrate the results of many similar runs by showing omni-directional magnetic energy spectra for two identically-driven runs with $B_0=0$ (Fig. 1) and $B_0=1.0$ (Fig. 2), respectively. In contrast to previous studies that have focused almost exclusively on the alpha parameter,¹² our em-

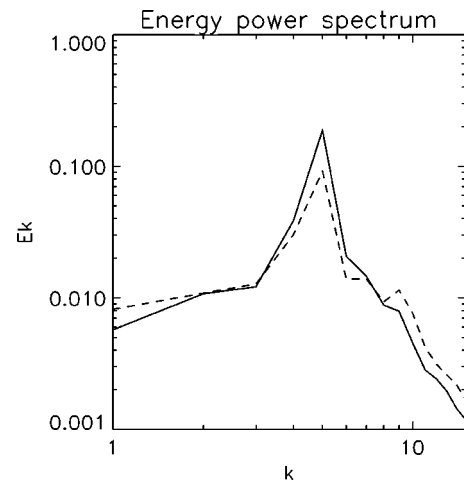


FIG. 2. Evolved ($t=1000$) kinetic energy spectrum (solid line) and magnetic energy spectrum (dashed line) in the presence of a dc magnetic field strength $B_0=1.0$ with helical mechanical forcing.

phasis will be on examination of the behavior of spectral quantities in response to dynamo action and the dc magnetic field strength. In particular the inverse cascade, if it occurs, involves transfer of magnetic energy to wavelengths longer than the mechanical forcing scales. The forcing function \mathbf{f} can be adjusted to inject mechanical excitations whose helicity varies between zero and the maximum possible for the chosen mean forcing amplitude. [To accomplish this, we select each Fourier mode $\mathbf{v}(\mathbf{k})$ with \mathbf{k} in the forcing band, and decompose into positive and negative mechanical helicity contributions (Chandrasekhar–Kendall functions).] Each real degree of freedom is given a random increment scaled to $F\Delta t(1+\sigma)/2$ for the positive helicity amplitude, or to $F\Delta t(1-\sigma)/2$ for negative helicity modes. Forcing strength is controlled by F (typically 1), while the helicity injection is regulated by σ . The timestep is Δt . For the runs shown in Figs. 1 and 2, $\sigma=0.8$ corresponds to roughly 80% of the injected energy going into positive helicity modes. The forcing band in these runs includes all wavenumbers between 5.0 and 5.3 in magnitude; a total of 67 independent complex vector amplitudes are driven. At the beginning, the energy spectra are both empty.

In Figs. 1 and 2 the solid lines are the mechanical energy spectra and the broken lines are the magnetic energy spectra, both at times $t=1000$. The unit of time is one eddy turnover time, based on unit length scale and unit rms velocity. The forcing amplitude $F=1$ is tuned so that the rms velocity field will be of order unity at the end of the run (saturation). The peaks in the spectra occur at the forcing band, where the mechanical excitations are being injected. At high wavenumber the magnetic energy exceeds the kinetic energy by a small amount, as is typical of MHD turbulence. Near the forcing wavenumber, kinetic energy dominates, reflecting the nature of the forcing. At wavenumbers lower than the forcing band, the spectra differ greatly for the two runs shown.

Figure 1, having $B_0=0$, essentially reproduces the results of Meneguzzi *et al.*⁴ It will be seen that the longest wavelength magnetic energy components have grown to more than an order of magnitude greater, in energy, than the velocity-field components that are driving them. The longest wavelength allowed by the boundary conditions dominates the spectrum. Everything below the forcing band in k-space represents spectral back-transfer to longer wavelengths. Figure 2, with $B_0=1$ on the other hand, shows no such accumulation of energy of either kind at the longest wavelengths, which appear to be dominated by low-amplitude Alfvén waves that imply near-equipartition of energy between the magnetic and kinetic spectra. The dynamo action at long wavelengths has been suppressed. Saturation has been achieved for both cases well before the time depicted.

Figures 3 and 4 show time histories of kinetic and magnetic energies for the same two runs in Figs. 1 and 2. For $B_0=0$, Fig. 3 shows that magnetic energy overtakes kinetic energy at about $t=75$ and thereafter the system as a whole is magnetically dominated. This is due to the strongly enhanced magnetic energy at the longest allowed wavelength as seen in Fig. 1. For $B_0=1$, Fig. 4 shows that magnetic energy saturates at a lower level, about 25%–30% less than the kinetic energy. In this “equipartition” regime, referring to Fig.

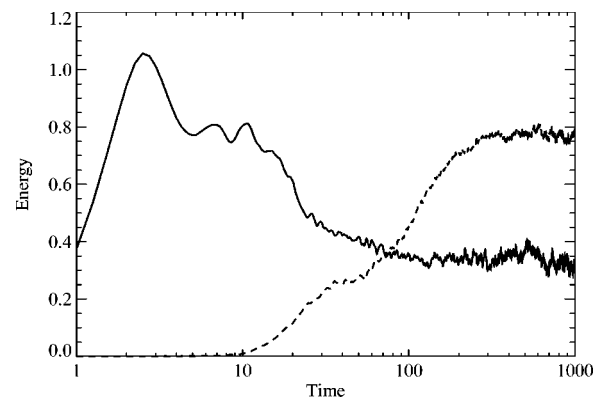


FIG. 3. Time histories of kinetic (solid line) and magnetic energy (dashed line) for the zero dc field case.

2, there is clearly no buildup of longest wavelength magnetic energy in the largest scale modes. Finally, we note that buildup of the magnetic energy, or the absence thereof, may be associated with the generation and long wavelength buildup of magnetic helicity, or its absence. This is evidenced by Fig. 5, showing normalized magnetic helicity and kinetic helicity spectra at $t=1000$ for $B_0=0$, and by Fig. 6, showing the same quantities $B_0=1$. For $B_0=0$ this conclusion is completely consistent with Meneguzzi *et al.*,⁴ except that we see it as a saturated effect and at much later times. Other runs (details not shown) have been done with nonhelical ($\sigma=0$) mechanical forcing, and dynamo action such as that seen in Figs. 1 and 5 is not observed. We have defined magnetic helicity here as the volume integral of $\mathbf{a} \cdot \mathbf{b}$, where the \mathbf{b} does not contain \mathbf{B}_0 , and is given by $\mathbf{b}=\nabla \times \mathbf{a}$. The mechanical helicity is defined as the volume integral of the dot product of velocity and vorticity, in the usual way.

In still other runs, we have explored the effect of lowering B_0 in the presence of a fixed helical random forcing to see when and if the dynamo action would reappear. We did find a threshold value, somewhere between $B_0=0.1$ and 0.03 in the dimensionless units, in which long-wavelength magnetic helicity-driven dynamo action reappeared. Below this value, the system evolved substantially as in Figs. 1 and 3, with a slight temporal offset. We do not hazard a guess as to what physical parameters the threshold may be dependent

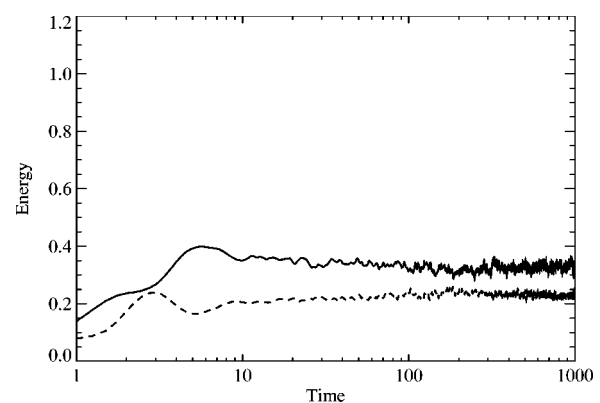


FIG. 4. Time histories of kinetic (solid line) and magnetic energy (dashed line) for the strong dc field case.

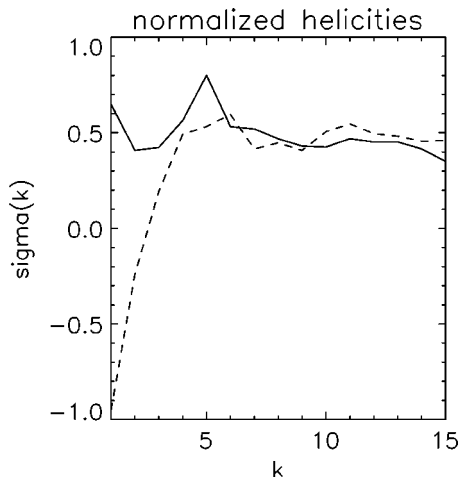


FIG. 5. Highly helical magnetic structure at the longest wavelength with $B_0=0$. Normalized kinetic and magnetic helicities for the zero dc magnetic field case at $t=1000$. In each case the associated gauge invariant normalized helicity is $(E^+ - E^-)/(E^+ + E^-)$ where E^+ and E^- are the decomposition of the respective energy into positive and negative helicity contributions in the relevant wavenumber range. Normalized helicities are nearly equipartitioned in forcing band and high wavenumber regimes. Note the very strong *negative* magnetic helicity at the longest wavelength. Combined with the magnetic energy spectrum in Fig. 1, this indicates a strong inverse cascade driven by magnetic helicity.

upon, because there are too many: ν , η , B_0 , the intensity of forcing, the location of the forcing band in k -space, and so on.

III. A RECONSIDERATION OF BOUNDARY CONDITIONS

There had previously been some unexplained but not widely noticed features of turbulent MHD behavior in 3D rectangular periodic boundary conditions that suggested less than a complete understanding of the role the geometry was playing. In an alpha effect calculation generalized to the case of a uniform dc magnetic field, Montgomery and Chen¹⁸ had found an amplification matrix whose trace tended to zero

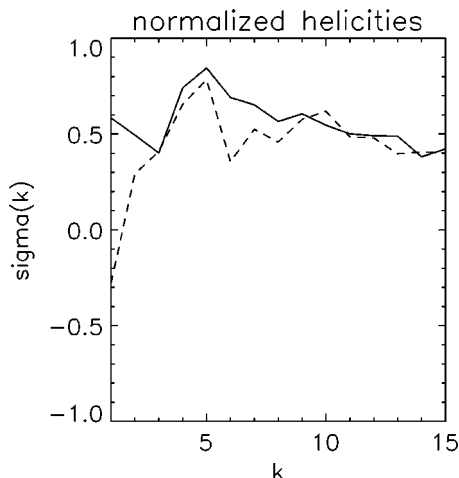


FIG. 6. Normalized kinetic and magnetic helicities for the strong dc magnetic field case at $t=1000$. There is little helicity of either type at the longest wavelength. There is no inverse cascade, and dynamo action is suppressed.

when B_0 became large, indicating a less and less efficient alpha amplification for larger and larger imposed dc magnetic field. A 3D periodic computation of decaying MHD turbulence in a dc magnetic field¹⁷ showed no tendency for long wavelength helical magnetic field components to persist in the fashion they would when B_0 was zero. Both behaviors are consistent with, though they do not imply, the behavior reported in Sec. II for the driven case.

Reluctantly, we have come to conclude that certain features of the combination of a dc magnetic field and rectangular periodic boundary conditions are unsatisfactory, and that these (computationally very convenient) boundary conditions, rather than the inherent physics, are controlling the computed behavior (see Ref. 19). In Ref. 19, reasons for distrusting triply-periodic boundary conditions as adequate for this problem were spelled out in considerable detail, and it has seemed unnecessary to reproduce them here. What seems to occur when the dc magnetic field is present and helical driving occurs is that the excitations built up are essentially Alfvén waves of a preferred helicity. A net emf builds up due to their attempts at dynamo action, corresponding to a nonzero spatially-averaged electric field parallel to \mathbf{B}_0 . In nature, this would seek to drive a net current along \mathbf{B}_0 , creating more magnetic flux in the perpendicular directions, but the rectangular periodic boundary conditions combined with Ampere's law permit no net current to flow through the system. "Open circuit" boundary conditions have been effectively imposed. In a physical plasma, what would result from this would be a migration of electrons to one face of the box (normal to \mathbf{B}_0) and a net positive charge would appear on the opposite face, screening out the mean interior electric field parallel to \mathbf{B}_0 . But that, too, is forbidden by the rectangular periodic boundary conditions.

Clearly there are limitations on the physics that can be represented by periodic boundary conditions, and care must be employed in using and interpreting them.^{17,19,12,13} It is necessary to recall that in electromagnetic theory, the theorems are for finite-sized systems whose fields fall off at infinity. "Infinite" systems are a convenient idealization, when considering, e.g., long current-carrying wires or parallel-plate capacitors, but it is still necessary to be able at least to imagine the infinite system as a limit of a finite one which becomes large. Similarly, the approximation of spatial homogeneity, often associated with the periodic model, is at best a local approximation. Moreover, attempts to extend homogeneity into the infinite domain limit may be fraught with difficulties, especially for MHD. The need for source currents for \mathbf{B}_0 somewhere, for each box in a triply-periodic array, clouds the picture of just what system it is that could be idealized by a periodic box that repeats itself indefinitely in all directions with a uniform dc magnetic field simultaneously present. Consequently, a model consisting exclusively of periodic cells is too idealized to represent the entirety of a physical system, especially one like a dynamo in which small scales must communicate dynamically with very large scales. There appear to be two possible solutions to this difficulty, which are related. First, the periodic system might be embedded in a larger system, using some version of multiple scale analysis to connect them.^{17,20} In such an approach,

periodic physics is a local effect and large scale dynamical processes such as dynamo action will be taking place in the “outer” model, itself not periodic. This approach, unless completed, cannot answer the question of whether dynamo action occurs, since the currents generated by the local $(\mathbf{v} \times \mathbf{b})$ lie outside the periodic model. Unless the full model is solved we do not know if such current could support amplification of the large scale field and at what level.

A more complete approach would be to formulate the dynamo problem in its entirety at the onset, in a framework that is not periodic. In the next section, we consider whether it may not be possible, by incorporating more realistic boundary conditions, to do dynamo computations that do not experience what seems to be an artificial suppression in the presence of a mean dc magnetic field.

IV. DISCUSSION AND FUTURE POSSIBILITIES

We should stress that our goal here is physical understanding of one likely dynamo process for attaining long-wavelength magnetic fields out of turbulent microscopic MHD processes. For that reason, we have not addressed ourselves to the many admirable efforts at specific simulations of solar or geophysical magnetic fields, which necessarily incorporate many processes and effects omitted here, and whose success or failure is generally judged by the simulations’ capacity for reproducing a wide range of observational features, such as sunspot patterns, solar prominences, periodicities, etc. We are focusing instead on only one MHD turbulent process in isolation, in an effort to understand it more correctly.

If triply-periodic rectangular boundary conditions are to be given up, it is natural to ask what other possible geometries and boundary conditions there are for asking fundamental questions about MHD dynamos. A first answer might be, spheres or disks¹⁴ for astrophysical situations, and toroids or periodic cylinders^{21–24} for laboratory ones. In all three cases, there is at least one coordinate, the radial one, which cannot be periodic. Turbulent inverse cascades of magnetic excitations in response to small-scale mechanical stimuli are readily imaginable in all three cases, though some symmetries that are convenient, such as isotropy and homogeneity, will be lost. Some experience has already been accumulated with weak MHD turbulence (though not for dynamo problems) in a rigid-wall, straight-cylinder geometry.^{21–24} Tractable boundary conditions for finite magnetofluids are non-trivial and need to be debated, but incorporating them seems a likely direction in which to proceed to try and formulate the proper problem. The three cases above, in the incompressible limit, all have natural expansion bases for Galerkin-method computations (Chandrasekhar–Kendall eigenfunctions of the curl) that suggest themselves and have been used to some effect in the past.

We conjecture that in more realistic geometries, finite in at least two dimensions, it will turn out that the presence of dc magnetic fields whose sources do not necessarily lie inside the magnetofluid do not act as a barrier to dynamo action of the inverse cascade type. That is, we do not expect long wavelength dynamo action in such geometries to be shut down by the presence of externally-supported dc magnetic fields. Only a formidable computational effort stands between us and a detailed answer. We reemphasize that what we are offering is a conjecture, and not a fact; and we stress again the need for driven turbulent MHD computations inside finite geometries with imposed dc magnetic to reinforce or refute the conjecture.

ACKNOWLEDGMENT

This research was supported in part by NSF Grant No. ATM-9713595.

- ¹U. Frisch, A. Pouquet, J. Leorat, and A. Mazure, *J. Fluid Mech.* **68**, 769 (1975).
- ²A. Pouquet, U. Frisch, and J. Leorat, *J. Fluid Mech.* **77**, 321 (1976).
- ³A. Pouquet, in *Proceedings of Les Houches Summer School on Astrophysical Fluid Dynamics*, edited by J. P. Jahn and J. Zinn-Justin (Elsevier, Amsterdam, 1987), pp. 139–227.
- ⁴M. Meneguzzi, U. Frisch, and A. Pouquet, *Phys. Rev. Lett.* **47**, 1060 (1981).
- ⁵H. K. Moffatt, in *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, 1978).
- ⁶F. Krause and K. H. Rädler, in *Mean-Field Magnetohydrodynamics and Dynamo Theory* (Clarendon, Oxford, 1980).
- ⁷G. B. Field, E. G. Blackman, and H. Chou, *Astrophys. J.* **513**, 638 (1999).
- ⁸J. V. Shebalin, W. H. Matthaeus, and D. Montgomery, *J. Plasma Phys.* **29**, 525 (1983).
- ⁹S. Oughton, E. Priest, and W. H. Matthaeus, *J. Fluid Mech.* **280**, 95 (1994).
- ¹⁰D. Montgomery and W. H. Matthaeus, *Astrophys. J.* **447**, 706 (1995).
- ¹¹L. J. Milano, W. H. Matthaeus, P. Dmitruk, and D. C. Montgomery, *Phys. Plasmas* **8**, 2673 (2001).
- ¹²F. Cattaneo and D. W. Hughes, *Phys. Rev. E* **54**, 4532 (1996).
- ¹³E. G. Blackman and G. B. Field, *Astrophys. J.* **534**, 984 (2000).
- ¹⁴A. Brandenburg and K. J. Donner, *Mon. Not. R. Astron. Soc.* **288**, L29 (1997).
- ¹⁵D. Gottlieb and S. A. Orszag, in *Numerical Analysis of Spectral Methods: Theory and Applications* (SIAM, Philadelphia, 1977).
- ¹⁶M. A. Berger, *J. Geophys. Res.* **102**, 2637 (1997).
- ¹⁷T. Stribling, W. H. Matthaeus, and S. Ghosh, *J. Geophys. Res.* **99**, 2567 (1994).
- ¹⁸D. Montgomery and H. Chen, *Plasma Phys. Controlled Fusion* **26**, 1189 (1984).
- ¹⁹D. C. Montgomery and J. W. Bates, *Phys. Plasmas* **6**, 2727 (1999).
- ²⁰W. H. Matthaeus, in *Geophysical Mono. 111. Proceedings of Magnetic Helicity in Space and Laboratory Plasmas*, edited by M. R. Brown, R. C. Canfield, and A. A. Pevtsov (American Geophysical Union, Washington, DC, 1999).
- ²¹X. Shan, D. Montgomery, and H. Chen, *Phys. Rev. A* **44**, 6800 (1991).
- ²²X. Shan and D. Montgomery, *Plasma Phys. Controlled Fusion* **35**, 619 (1993).
- ²³X. Shan and D. Montgomery, *Plasma Phys. Controlled Fusion* **35**, 1019 (1993).
- ²⁴X. Shan and D. Montgomery, *Phys. Rev. Lett.* **73**, 1624 (1994).