Conditions for sustainment of magnetohydrodynamic turbulence driven by Alfvén waves*

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In a number of space and astrophysical plasmas, turbulence is driven by the supply of wave energy. In the context of incompressible magnetohydrodynamics (MHD) there are basic physical reasons, associated with conservation of cross helicity, why this kind of driving may be ineffective in sustaining turbulence. Here an investigation is made into some basic requirements for sustaining steady turbulence and dissipation in the context of incompressible MHD in a weakly inhomogeneous open field line region, driven by the supply of unidirectionally propagating waves at a boundary. While such wave driving cannot alone sustain turbulence, the addition of reflection permits sustainment. Another sustainment issue is the action of the nonpropagating or quasi-two dimensional part of the spectrum; this is particularly important in setting up a steady cascade. Thus, details of the wave boundary conditions also affect the ease of sustaining a cascade. Supply of a broadband spectrum of waves can overcome the latter difficulty but not the former, that is, the need for reflections. Implications for coronal heating and other astrophysical applications, as well as simulations, are suggested. [DOI: 10.1063/1.1344563]

I. INTRODUCTION

A number of space and astrophysical plasmas engage in vigorous heating, transport, and diffusion processes associated with dynamically active fluctuations. Often this activity can be described by the equations of magnetohydrodynamics (MHD). It is not an infrequent circumstance that turbulence is driven by an input of fluctuation energy in the form of waves. In the incompressible or nearly incompressible limit such waves would be described as Alfvén waves. Wave energy might be supplied as a flux through a boundary. This would correspond, for example in a solar context, to waves launched upward from the chromospheric network region. After propagation into the lower corona, the dissipation of such waves, possibly through a cascade process, may be associated with observed intense heating.[1] A somewhat distinct scenario is that in which wave particle interactions,[2] acting as a body force, generate large amplitude Alfvén waves. This viewpoint would be relevant to turbulence and fluctuations near comets, or in the outer heliosphere where newly ionized interstellar charged particles scatter and excite Alfvén waves. There is some evidence that this process enhances cascade and dissipation processes.[3]

While details vary considerably, there is a common thread in these various circumstances that is of interest from the point of view of MHD turbulence theory: Turbulence is driven, not by random ''stirring'' of the velocity field as is often envisioned in hydrodynamics contexts, but rather by the injection of wave energy, either at a boundary, or as a volume force. Our purpose in this paper is to address the question of whether such wave driving can sustain MHD turbulence. By avoiding specific details associated with particular applications, and by adopting several idealizations, we can state the problem as an academic one in MHD turbulence theory: *Under what circumstances can one sustain incompressible MHD turbulence by the supply of energy in the form of unidirectionally propagating Alfvén waves?* Below we discuss some general conditions under which turbulence can be sustained in this way. We assume that the plasma is either homogeneous or weakly inhomogeneous, and is threaded by a strong uniform magnetic field. The ''top'' of the region of interest has ''open'' field lines, so waves can transport energy rapidly through the region. Our presentation is based upon physical properties of MHD, as supported by direct simulations.

II. RMHD MODEL: TURBULENCE WITH OPEN BOUNDARIES SUBJECT TO A MEAN MAGNETIC FIELD

To adopt a simple model that demonstrates the essential physics of interest, we consider a reduced MHD model (RMHD), appropriate to the low-frequency dynamics of an incompressible or weakly compressible plasma in the presence of a strong uniform constant magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$.[4–6] This one-fluid MHD model involves a fluctuating fluid velocity $\mathbf{v}(x,y,z,t) = (v_x, v_y, 0)$, a magnetic field $\mathbf{B}(x,y,z,t) = \mathbf{B}_0 + \mathbf{b} = (b_x, b_y, B_0)$, and a uniform constant density ρ. The magnetic field is expressed in Alfvén speed units. Large-scale Reynolds number R and magnetic Reynolds number R_m are reciprocals, respectively, of a uniform constant scalar viscosity μ and resistivity η.

Inherent to RMHD is the condition that gradients (of all variables) in the direction parallel to $\mathbf{B}_0$ are much weaker

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than those in the perpendicular directions. Thus, for example, $\partial_z \ll \partial_x$. Supplemented by the solenoidality conditions $\nabla \cdot \mathbf{b} = \nabla \cdot \mathbf{v} = 0$, this implies that the leading-order fluctuations can be described by the vorticity $\omega(x,y,\epsilon z)\hat{\mathbf{z}} = \nabla_\perp \times \mathbf{v}$ and the vertical component of the vector potential $a(x,y,\epsilon z)$ where $\mathbf{b} = \nabla_\perp \times a\hat{\mathbf{z}}$. Here we place a small parameter ϵ in front of the z coordinate to emphasize the anisotropy of the RMHD representation. Hereafter we denote the slowly-varying coordinate along the mean field by s. The appearance of the perpendicular gradient $\nabla_\perp = (\partial_x, \partial_y, 0)$ is also a manifestation of this anisotropy. Preferential spectral transfer to a high perpendicular wavenumber[7–10] is a well known feature of fully three dimensional (3D) MHD which favors dynamical generation of the anisotropic conditions described by RMHD.

Note that the electric current density is $j\hat{\mathbf{z}} = \nabla_\perp \times \mathbf{b}$ and the stream function ψ satisfies $\nabla_\perp^2 \psi = -\omega$. The dynamical equations of RMHD can now be written in terms of the Elsässer variables, $\mathbf{z}_\pm = \mathbf{v} \pm \mathbf{b}$,

$$\frac{\partial \mathbf{z}_\pm}{\partial t} \mp V_\mathrm{A} \frac{\partial \mathbf{z}_\pm}{\partial s} = -\nabla_\perp p' - \mathbf{z}_\mp \cdot \nabla_\perp \mathbf{z}_\pm + \eta \nabla_\perp^2 \mathbf{z}_\pm, \qquad (1)$$

where the total pressure $p' = p/\rho + B^2/2$ and we assumed $\mu = \eta$.

In the standard set of units in which the equations are written, the speed is measured in units of the typical turbulent fluctuation strength δv. The perpendicular length scales are in units of L, while the parallel length scale is in units of $L_z \gg L$, corresponding to the scale inequalities inherent in derivations of RMHD. By virtue of the same principle, the advective and Alfvén time scales are comparable, $L/\delta v \sim L_z/V_\mathrm{A}$, since $V_\mathrm{A} \gg \delta v$ while $L_z \gg L$.

To address basic questions regarding the feasibility of supporting steady MHD turbulence by Alfvén wave driving, we consider a region of space that is periodic in the x,y plane, but is bounded in the s direction with $s = [0,1]$ in appropriate dimensionless units. Wave flux is supplied at the lower boundary, $s = 0$. It is convenient to employ the Elsässer potentials $f = \psi - a$ and $g = \psi + a$. These are the potentials associated with the variables $\mathbf{z}_\mp = \mathbf{v} \mp \mathbf{b}$ which correspond to negative and positive cross helicity fluctuations, respectively. Note that for $B_0 > 0$, i.e., an upward directed mean field, a packet of purely upward propagating waves is a packet of f, while a downward propagating packet is described by g. However, the quasi-two dimensional or ''zero-frequency'' modes of the system, described by $\partial_s \mathbf{v}_{2D} = 0 = \partial_s \mathbf{b}_{2D}$, may have either positive or negative cross helicity. These are the nonpropagating modes, whose importance will be emphasized below. For any f and g, the nonpropagating parts may be extracted by averaging in s, e.g., $\mathbf{v}_{2D} = 1/2 \int \nabla_\perp \times \langle f - g \rangle_s \hat{\mathbf{s}} ds$.

It is straightforward to apply the desired boundary conditions at $s = 0$ and $s = 1$ in terms of the potentials f and g, which are characteristic variables for the wave part of Eqs. (1). We need not control the potentials on outward propagating characteristics since these are intended to allow upward waves to escape at the top and downward waves to escape at the bottom. Thus $g(x,y,s=0,t)$ and $f(x,y,s=1,t)$ are left uncontrolled. We wish to control the wave flux entering at the base, which is set to a known function. We also want to set to zero the wave flux entering through the top. These are accomplished in one of two ways. First we may choose to control the corresponding potentials themselves. This is done by setting $f(x,y,s=0,t) = F_0(x,y,t)$ and setting $g(x,y,s=1,t) = 0$. The second method is to control the spatial derivatives of the potentials. That is, at the bottom we specify $\partial_s f(x,y,s=0,t) = \hat{F}_0(x,y,t)$ for some appropriately chosen function $\hat{F}_0$, while at the top we choose $\partial_s g(x,y,s=1,t) = 0$.

For brevity we refer hereafter to these two possibilities as the $g = 0$, $f = F_0$ case (fixed potential), or the $\partial_s g = 0$, $\partial_s = \hat{F}_0$ case (fixed derivative).

The (different) effects of these distinct constraints upon the turbulence are discussed below.

Our numerical method involves solving Eqs. (1) via a Fourier pseudospectral treatment of the x and y variables. The wave operator terms in the coordinate s are handled by Chebyshev collocation, which facilitates imposing boundary conditions at $s = 0$ and $s = 1$ while providing high-order accuracy. Time is advanced using a second-order Runge–Kutta method.

The same strong mean field condition that favors development of anisotropy also induces the propagation of Alfvén waves along $\mathbf{B}_0$. Since we want to investigate the interplay between an imposed wave flux and MHD turbulence, the present RMHD framework thus appears to be ideal. A full MHD model would permit driving by very high frequency waves which fall outside of the RMHD restrictions. However such high-frequency fluctuations would be strongly *non*resonant with low-frequency energy-containing Fourier modes, and would thus be ineffective in driving turbulence. The lower-frequency Alfvén waves retained in the RMHD model have a better chance of sustaining turbulence. Moreover, the RMHD model provides a simple characteristic structure and an efficient and easily understood representation which facilitates the making of our basic points.

III. UNIDIRECTIONAL WAVES ALONE CANNOT SUSTAIN TURBULENCE

The first point we make is one that is based upon ideas already well documented in the literature.[11–15] The question at hand is whether turbulence can be sustained by forcing consisting solely of unidirectionally propagating waves. Although the answer is negative, and is a corollary of prior studies (see below), we will demonstrate this first numerically, in part to elucidate our numerical approach.

The simulations shown here, labeled Runs (I), (II), (III), and (IV), employ periodic boundaries in x and y, and an open wave propagation scheme in the s direction, as described in the prior section. We have studied various definitions for the imposed wave flux at the bottom boundary, F_0 $(x,y,s=0,t)$. However for the several cases illustrated here, we will restrict ourselves to monochromatic forcing, by which we mean the following. We may define the perpendicular spectral decomposition of the potentials as $f(x,y,z,t) = \Sigma_{\mathbf{k}_\perp} \tilde{f}(k_x,k_y,s,t) e^{i(k_x x + k_y y)}$ and $g(x,y,s,t)$

$=\Sigma_{\mathbf{k}_\perp}\tilde{g}(k_x,k_y,s,t)e^{i(k_x x+k_y y)}$. Correspondingly the spectral structure of the lower boundary condition may be represented as $\tilde{F}_0(k_x,k_y,t)=\tilde{f}(k_x,k_y,s=0,t)$, which can be specified for fixed potential boundary conditions. Alternatively, for fixed derivative boundary conditions, the spectral structure will enter through specification of $\hat{F}_0(k_x,k_y,t)=\partial_s\tilde{f}(k_x,k_y,s=0,t)$. By monochromatic driving we mean that the transverse structure of the driven mode consists of a single perpendicular wave vector, while the time dependence is at a single frequency ν. Thus for monochromatic driving at transverse wavevector $(k_x,k_y)=(1,1)$ we choose, for fixed potential boundaries,

$$\tilde{F}_0(k_x,k_y,t)=\begin{cases} A\sin 2\pi\nu t, & \text{if } \mathbf{k}_\perp=(1,1),\\ 0, & \text{otherwise.}\end{cases} \tag{2}$$

For fixed derivative boundary conditions we choose

$$\hat{F}_0(k_x,k_y,t)=\begin{cases} -\dfrac{A2\pi\nu}{V_\mathrm{A}}\cos 2\pi\nu t, & \text{if } \mathbf{k}_\perp=(1,1),\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

It is clear by examination of the linear part of Eqs. (1) that the latter choice enforces a specified upward propagating wave flux at the bottom boundary, and in this respect accomplishes a wave driving similar to that of the fixed potential case, Eq. (2). We considered a low-frequency forcing throughout all the runs, with $\nu=0.1/t_\mathrm{A}$, where t_A is the Alfvén wave crossing time along direction s of the domain. For the top boundary, we require that the flux of downwards waves is zero, the latter being accomplished using either the $g=0$ or $\partial_s g=0$ boundary condition.

The initial conditions for the fields within the box are random spectra of fluctuations which are band-limited so that excited modes have $2\leqslant k_\perp\leqslant 6$. The cross helicity of the initial population is nearly zero. The question is whether the forcing can sustain this turbulence level against dissipation and losses due to propagation out of the simulation domain, or if, alternatively, the turbulence dies out.

Run (I) addresses this question employing the $g=0$ boundary condition. Figure 1 shows the results of a run with $x\times y\times s$ resolution of $64\times 64\times 31$. The top panel shows a time history of normalized cross helicity,

$$\sigma_c=\frac{\int\rho(z_+^2-z_-^2)\mathrm{d}x\mathrm{d}y\mathrm{d}s}{\int\rho(z_+^2+z_-^2)\mathrm{d}x\mathrm{d}y\mathrm{d}s}. \tag{4}$$

The middle panel shows, as a function of time, the ratio of the dissipation rate to the time averaged input of wave energy though the lower boundary. This is called the dissipation efficiency. (The average is taken over an oscillation period at the lower boundary.) The bottom panel is the fraction of total dissipation due to all Fourier modes other than the driven mode. This is a measure of nonlinear spectral transfer. For a simple wave that does not couple to other degrees of freedom, this quantity is zero; if it approaches unity it signifies that nearly all the excitations have transferred out of the driven mode. Transfer into smaller scale modes is emphasized.

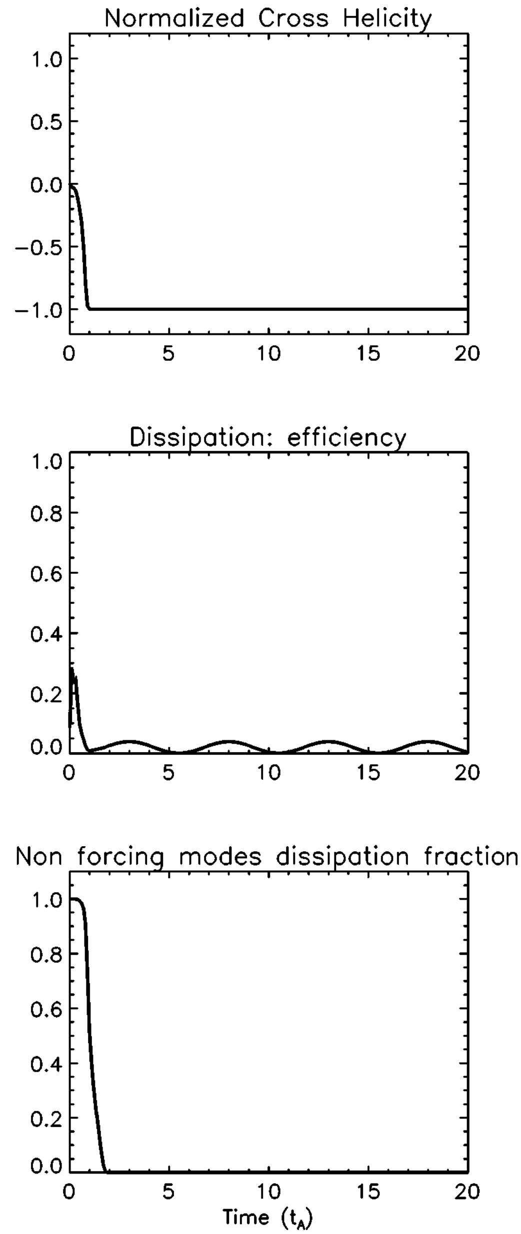

FIG. 1. Results from Run (I), which has a uniform Alfvén speed and upper boundary condition $g=0$, showing that turbulence is not sustained by a unidirectionally propagating wave flux supplied at the lower boundary in an open field configuration. Top panel: Normalized cross helicity σ_c (see the text) as a function of time. After a rapid decrease over a time scale of the order of the Alfvén crossing time, $\sigma_c\rightarrow -1$ indicating pure upward propagation. Middle panel: Dissipation efficiency (ratio of dissipation to period-averaged input wave energy flux) vs time. The low and oscillatory level indicates simple periodic dissipation associated with a single propagating wave. Bottom panel: Ratio of total dissipation in all undriven modes, to dissipation in the driven mode. Again the decrease to ≈ 0 after an Alfvén time suggests there is little or no spectral transfer after that time. The turbulence has died out and is replaced with a simple propagating wave.

Referring to the results for Run (I) in Fig. 1, we see that the normalized cross helicity rapidly decreases from its (arbitrary) initial value of zero to values very close to $\sigma_c=-1$ by $t\approx 1$, which is one Alfvén crossing time of the s dimension of the domain. The dissipation efficiency also decreases rapidly (after a brief transient growth), and then bounces between zero and a small value in a regular way.

The behavior shown by the final diagnostic indicates that the dissipation is initially due to interactions of many modes, but for $t>1$ nearly all dissipation is associated with direct damping of the driven mode. The conclusion is clear—the initially present turbulence is responsible for a burst of dissipation early in the run, but it vanishes in an Alfvén time, and is replaced by what is essentially a pure upward traveling wave. This is a representative result.

Although the path taken is somewhat differently, similar results are obtained in Run (II), for which the parameters of Run (I) are adopted, aside from changing the boundary condition from fixed potential conditions to fixed derivative conditions.

Results of Run (II) are summarized in Fig. 2. The top and middle panels are very similar to Fig. 1, showing that the fluctuations are dominated by upward propagating waves after about $t=4$. There is evidence of some nonlinear activity at later times associated with the persistent wobbles in the dissipation efficiency curve. This is borne out in the lower panel, which clearly shows that the dissipation due to nonforced modes recurs for a much longer time than in Run (I). The boundary conditions have made a manifest difference. Nonetheless, turbulence is still not sustained, although it takes much longer for it to die out completely.

Note that both boundary conditions disallow entry of a downward wave flux (i.e., propagating modes) from the top. The difference between the two cases is that $g=0$ [Run (I)] eliminates, in addition, any nonpropagating mode amplitude after one Alfvén time. In contrast, $\partial_s g=0$ [Run (II)] does not eliminate the nonpropagating modes, which can appear in the initial data, and are left to freely relax due to nonlinear effects and dissipation.

To understand further this point, we can consider the detailed balance of upward and downward fluctuation energies at each plane $s=$const. From the dynamical equations (1), the equations for the fluctuation energies at each perpendicular plane are obtained by integrating in (x,y),

$$\frac{\partial\langle|z_\pm|^2\rangle}{\partial t}=\pm V_A\frac{\partial\langle|z_\pm|^2\rangle}{\partial s}-D_\pm, \tag{5}$$

where $\langle\cdots\rangle$ means an integral over (x,y) and $D_\pm$ are the dissipation terms. The nonlinear terms and the pressure terms do not contribute to the total energy balance at each plane, a result that can be readily seen from the dynamical equations (1) using the transverse periodicity of the fields. Dissipative terms can be shown to be single signed. The downward and upward fluctuation energies satisfy (aside from dissipation) wave equations which, more importantly, are completely decoupled from one another. It is clear that if $\langle|z_+|^2\rangle^{\rm top}=0$ [as imposed by the $g=0$ boundary condition of Run (I)], then this condition will propagate through the box from the top and after one Alfvén time there will be no downward type of fluctuations. At this point, the nonlinear couplings (which necessary involve both $\mathbf{z}_+$ and $\mathbf{z}_-$) will be no longer possible and the initial level of turbulence dies. In the case of Run (II), the condition $\partial_s g=0$ does not impose $\langle|z_+|^2\rangle^{\rm top}=0$, but instead it implies that $\partial_t\langle|z_+|^2\rangle^{\rm top}=-D_+^{\rm top}$, so that the $\mathbf{z}_+$ energy at the top boundary relaxes in a dissipation time,

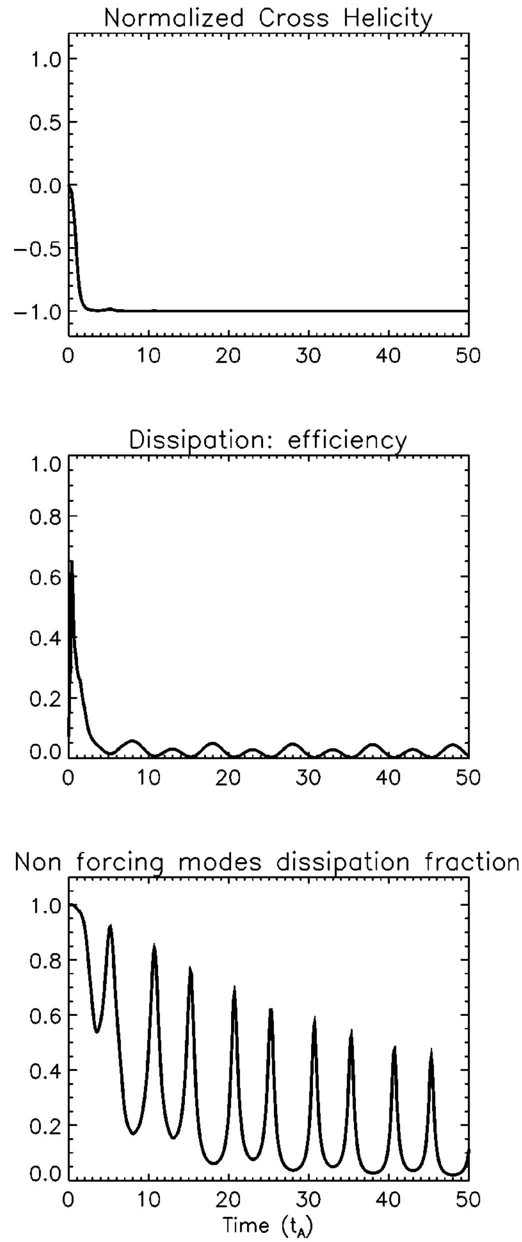

FIG. 2. Results from Run (II), which differs from Run (I) only in the boundary condition at the top, here fixed derivative conditions are imposed [e.g., $\partial_s g(s=1)=0$], which allows nonpropagating modes. The format is as in Fig. 1. Normalized cross helicity and dissipation efficiency again tend towards zero, but more slowly than in Run (I). The fraction of dissipation taking place in the undriven modes displays large oscillations but also tends to zero. Evidently turbulence cannot be sustained in this situation either.

which is much longer than the Alfvén time. When the nonpropagating modes are permitted in the simulation [Run (II)], positive cross helicity $(\mathbf{z}_+)$ modes can persist beyond the "emptying time" of Run (I). It is well established [8,9,16] that nonpropagating modes act as an efficient catalyst for perpendicular spectral transfer. Consequently the observation that the turbulence takes much longer to die out in Run (II) can be understood as a direct consequence of the existence of nonlinear couplings involving the nonpropagating fluctuations.

In spite of the interesting differences between Runs (I) and (II), both support the conclusion that turbulence cannot be regenerated and sustained by driving due to a *uni*directionally propagating monochromatic Alfvén wave in a homogeneous plasma. These conclusions are actually a corollary of a significantly stronger statement: *Turbulence cannot be sustained by unidirectionally propagating waves even if the driven wave spectrum is broadband.* The key to understanding this is to realize that driving by unidirectionally propagating waves supplies energy to one of the Elsässer variables, but not the other, as it can be seen from the decoupled energy equations (5). If there is no supply of the downward type of fluctuations, then, those fluctuations will disappear through dissipation and/or transmission through the bottom boundary. At that moment, no turbulence is possible since the nonlinear terms, which necessarily involve both $\mathbf{z}_+$ and $\mathbf{z}_-$, will be zero.

IV. WAVES AND TURBULENCE IN A SMOOTHLY INHOMOGENEOUS MEDIUM

Some additional effect has to be included to have any possibility of sustaining turbulence with the type of driving employed here. In most of the applications we have in mind in space and astrophysics, the large-scale magnetic field and the density are not uniform as we considered them to be above. It is well known that waves propagating in an inhomogeneous medium can be reflected due to gradients in their propagation speed. A number of authors have extensively studied the influence of inhomogeneities on the linear propagation of Alfvén waves mostly in the context of the interplanetary medium or model solar atmospheres.[17–23] General frameworks for transport locally incompressible turbulence in weakly inhomogeneous media, including inhomogeneous flows and nonlinear effects, have been developed.[24–27] In all cases one finds that large-scale spatial variations in the magnetic field and density cause reflection. Even a relatively small amount of reflection of an imposed propagating wave train raises the possibility that counter-propagating wave trains can interact and excite turbulence. This ''mixing'' effect has been suggested to play a role in triggering turbulence in both the solar wind[27] and in the open field line corona.[28] Here we ask whether this effect has the sought after influence of permitting the maintenance of steady turbulence using waves as the forcing mechanism.

In the simplest case reflection can be introduced by allowing for weak inhomogeneity along the direction of the magnetic field. This implies that $V_A = V_A(s)$. For the present model, let us consider a normalized Alfvén velocity profile, $V_A(s)$,

$$V_A(s) = [1 + 3(s - \pi^{-1}\cos(\pi s)\sin(\pi s))], \tag{6}$$

with s in the interval $[0,1]$. This quantity starts at $V_A\ (s=0)=1$ and smoothly varies up to $V_A\ (s=1)=4$. The modified RMHD equations in an inhomogeneous medium are (neglecting a possible mean flow speed and using planar geometry),

$$\frac{\partial \mathbf{z}_\pm}{\partial t} \mp V_A \frac{\partial \mathbf{z}_\pm}{\partial s} = \mp \frac{1}{2}\frac{dV_A}{ds}\mathbf{z}_\pm \pm \frac{1}{2}\frac{dV_A}{ds}\mathbf{z}_\mp, \\ -\nabla_\perp p' - \mathbf{z}_\mp \cdot \nabla_\perp \mathbf{z}_\pm + \eta \nabla_\perp^2 \mathbf{z}_\pm . \tag{7}$$

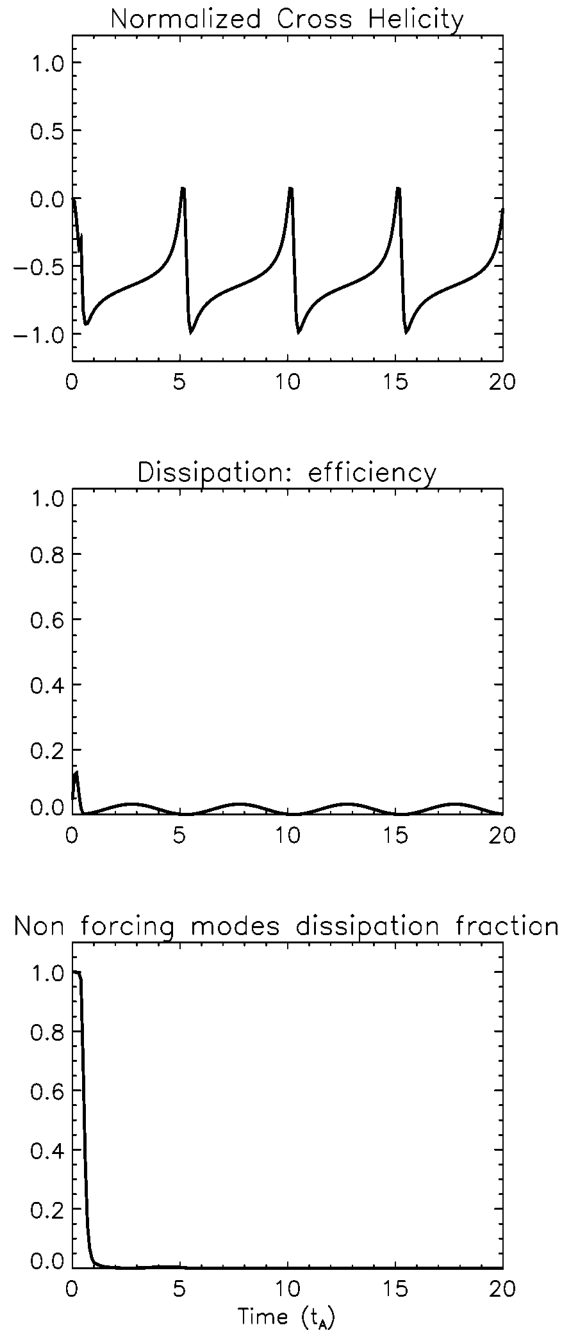

FIG. 3. Results from Run (III), which includes the effects of reflection associated with inhomogeneous Alfvén speed. As in Run (I), the boundary condition at the top, $g\ (s=1)=0$, does not allow nonpropagating modes. The oscillation of normalized cross helicity and dissipation efficiency indicate recurrent activity but very little spectral transfer. The fraction of dissipation in undriven modes is small except at early times. After that point, there is no suggestion of turbulence.

We now ask whether our modified problem permits the sustainment of turbulence. In Run (III) we computed the solutions to the same problem as in Run (I), but with reflection included as above. Note that the upper boundary condition is $g=0$, so the nonpropagating modes are excluded. Figure 3 summarizes the results of this simulation, which was carried out using the same methods as in the previous section [Chebyshev collocation easily permits the extension to nonuniform $V_A(s)$]. The upper panel of Fig. 3 shows that a large persistent oscillation of normalized cross helicity σ_c is set up. The system periodically achieves states of almost purely outward propagating fluctuations, and alternately states with nearly equal mixtures of inward and outward fluctuations. The dissipation efficiency oscillates but remains very low, suggesting little spectral transfer. This is confirmed in the bottom panel. There is a brief period early in the run during which there is significant dissipation in undriven modes due to decay of the initially present turbulence. Subsequently, however, there is no discernible dissipation except in the driven mode. Again, turbulence has not been sustained, this time in spite of moderately strong reflection.

Run (IV) repeats Run (III), but with the boundary conditions changed to fixed derivative boundaries (e.g., $\partial_s g = 0$) (Fig. 4). Now nonpropagating modes are permitted. Normalized cross helicity once again increases in magnitude as it did in the previous cases, due to supply of upward traveling modes at the lower boundary. However in this instance, a statistically steady plateau, away from the extremal value of -1, is attained after $t \approx 5$.

This simulation has been extended to longer times ($t = 100$) than the previous ones, to effectively show that the turbulence is sustained. Dissipation efficiency (Fig. 4, middle panel), increases and does not return to near-zero values as it did in all other runs; it does exhibit large pulsations associated with the periodic monochromatic driving. The bottom panel of Fig. 4 shows that the fraction of dissipation occurring in the undriven modes remains at a very high level—near unity, in fact. It appears that Run (IV) has established a sustained level of statistically steady turbulence. This can be confirmed by the examination of a sequence of one dimensional energy spectra, computed as functions of transverse wavenumber $k_\perp$. Such a sequence is shown in Fig. 5, for $t = 0.0, 0.5, 2.5$. This case corresponds to a high perpendicular resolution version of Run IV, with $512 \times 512 \times 9$ grid-points in $x \times y \times s$, but keeping the same parameters and boundary conditions as in Run IV. The solid line suggests a spectral slope that is approximately $k_\perp^{-5/3}$ as would be expected for steady driven MHD turbulence.

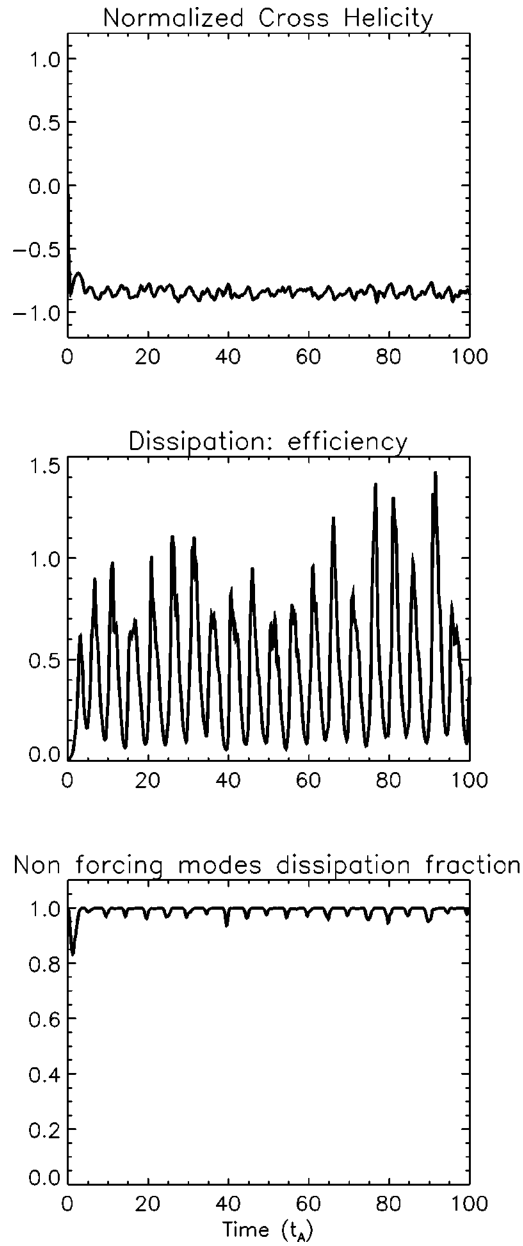

FIG. 4. Results from Run (IV), which includes reflection effects and has fixed derivative boundary conditions [e.g., $\partial_s g(s=1)=0$], permitting non-propagating modes. Normalized cross helicity quickly attains a steady value, $\sigma_c \approx -0.8$. Dissipation efficiency varies quasi-periodically but remains above ~ 0.1 with average ≈ 0.45. Fraction of dissipation in undriven modes remains high (near unity). MHD turbulence is sustained in this case.

V. DISCUSSION AND CONCLUSIONS

We have examined a series of simulations of RMHD to address the question of whether MHD turbulence can be driven and sustained by Alfvén wave driving alone. The focus has been upon the supply of unidirectionally propagating Alfvén waves that are monochromatic in the sense that a single transverse wave vector is driven at the ''lower'' boundary, and at a single low frequency.

Thus in an infinite domain and in the absence of reflection or nonlinearity, a propagating nondispersive Alfvén wave would be driven at a well defined wave vector and frequency. Two upper boundary conditions were employed, both of which permit no entry of downwards waves from the top boundary. However one choice ($\partial_s g = 0$) permits nonpropagating modes to be present while the other ($g = 0$) eliminates them. Our conclusion is that turbulence cannot be sustained for the homogeneous problem with no reflection, regardless of which boundary condition is employed. This can be understood easily because counter-propagating fluctuations, or, at least, mixed cross helicities, are required for incompressible MHD turbulence, as is well known. This theorem must be modified for a weakly inhomogeneous case in which reflection is present due to nonuniform Alfvén speed. Incorporating reflection, we show that turbulence once again cannot be sustained when the nonpropagating modes are excluded, but that it can be sustained when they are not excluded.

The above conclusions provide a firm answer to the restricted question that we posed. Applicability to more general situations can also be addressed to some degree. For

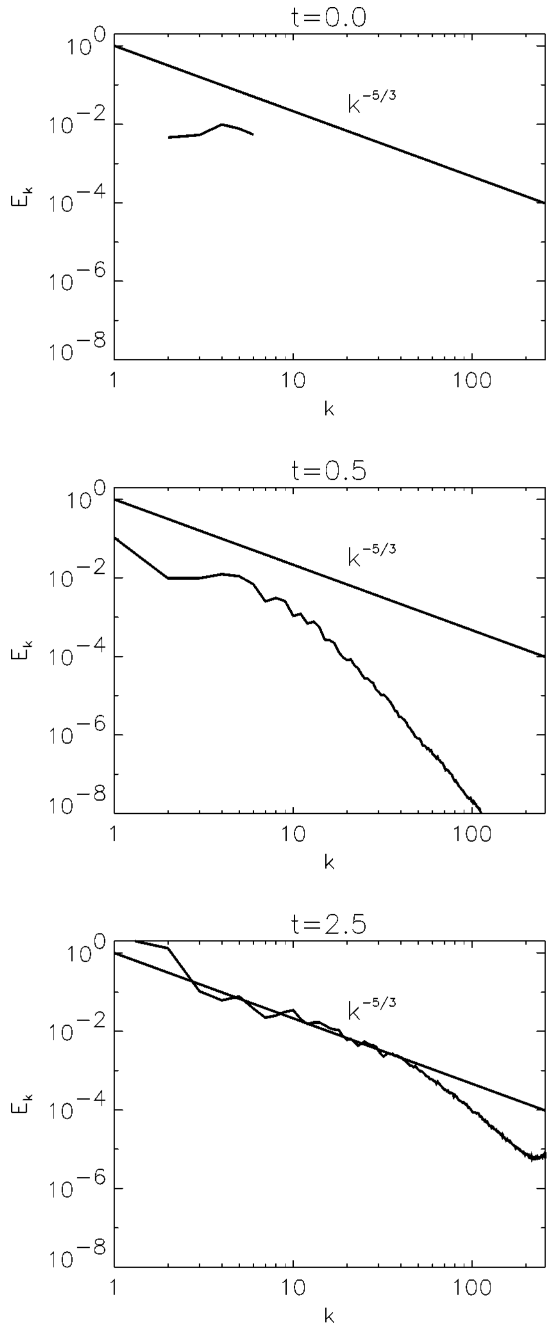

FIG. 5. Energy spectra from Run (IV) at three times, $t=0.0,0.5,2.5$. The straight line is $\sim k_\perp^{-5/3}$. A fully developed turbulence spectrum emerges from a seed level of turbulence, driven by a monochromatic wave in the presence of reflection due to Alfvén speed inhomogeneity.

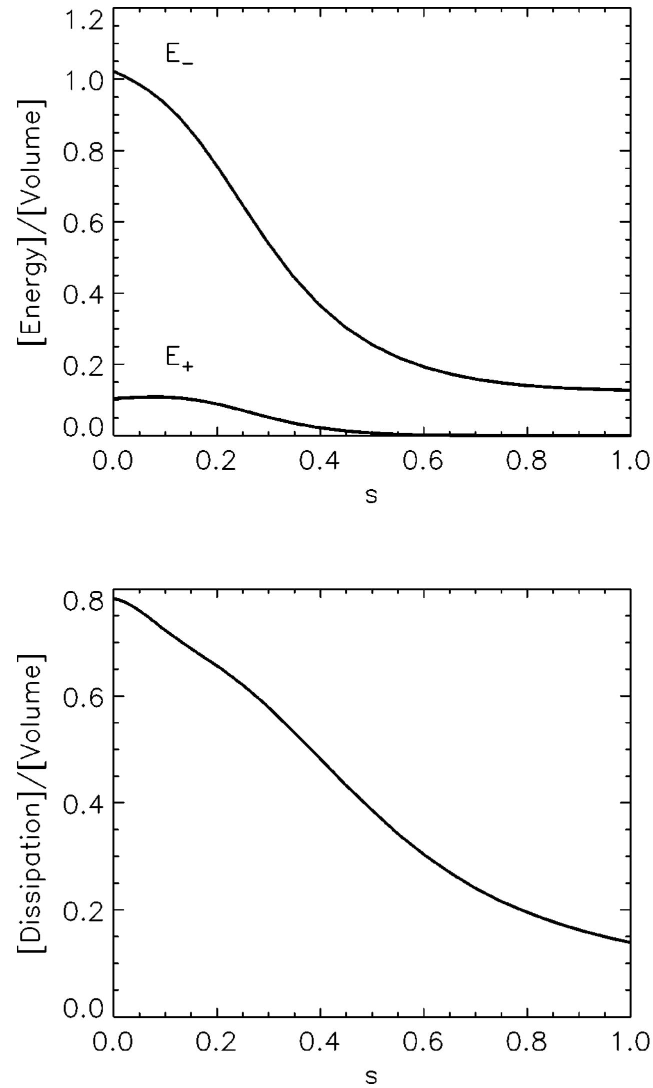

FIG. 6. Dependence of the solution on the vertical coordinate s, for the inhomogeneous Run (IV). Top panel: Energy distribution per unit volume (E_- for upward-type fluctuations and E_+ for downward-type fluctuations) as function of s and averaged in time over several forcing periods. Lower panel: Dissipation distribution per unit volume as function of s, averaged in time. The energy of downward-type fluctuations is comparatively higher on the lower region (close to $s=0$) than at the top ($s=1$) and dissipation is enhanced there.

example, what happens for wave driving that is broadband in wavenumber? Qualitatively we would expect that broadband forcing would immediately supply nonlinear couplings, without the need for a preliminary step of transferring excitation out of the directly supplied mode(s). Thus, we anticipate that broadband driving (with, say, a $k_\perp^{-5/3}$ transverse spectrum) would make turbulence easier to maintain, even in cases in which the nonpropagating modes are excluded by boundary conditions. Several runs of this type were carried out (not shown) and this reasoning was indeed confirmed. However, there is still an absolute requirement that reflection is present to sustain turbulence in any case in which the broadband forcing supplies only upward propagating fluctuations. The requirement of including nonpropagating fluctuations that we saw in our numerical results is evidently connected, in the monochromatic case, to the need for efficient first-step couplings that set up more numerous couplings required for a full cascade. With broadband forcing, these efficient or resonant first couplings are still helpful, but are not required.

Another broader circumstance of interest is one that results from relaxing the incompressibility constraint. Even for weak compressibility there may be channels for driving nonlinear couplings that are not present in the current discussion of incompressible MHD. These effects, which can be important when the main field $\mathbf{B_0}$ is not strong (or equivalently, for plasma $\beta \sim 1$), are considered in Ref. 29. Also, nonlinear wave equation formalisms, such as DNLS and its kinetic modifications,[30,31] might be able to enhance cascade effects in a strong compressible situation. Such considerations have been outside the present context, but remain as interesting topics for further research.

A final feature that we remark upon is the spatially nonuniform distribution of turbulence and dissipation in the inhomogeneous case Run (IV), in which steady turbulence was achieved through wave driving. As argued above, MHD turbulence requires spatial overlap of the upward-type ($\mathbf{z}_-$) and downward-type ($\mathbf{z}_+$) fluctuations. Upward waves are supplied at the lower boundary, and downward fluctuations will be generated at a rate controlled by the background Alfvén speed profile. The net result is that the dissipation is greatly enhanced in the lower part of the simulation domain. This is illustrated in Fig. 6 which shows both the (time averaged) distribution of fluctuation energies per unit volume ($E_\mp = \rho z_\mp^2$) and dissipation per unit volume, as functions of the vertical coordinate s. This has immediate consequences for applications such as solar coronal heating in which the waveflux enters from a more or less prescribed boundary. In such cases, it is to be expected that the heating rate due to turbulence will be confined, in a relative sense, to a region established by the reflection profile, which is itself determined by the Alfvén speed profile.

This provides a direct link between the large-scale field structure and the deposition of turbulent energy as heat, which is of clear importance in the problem of coronal heating. A further discussion of the model studied here in the context of coronal heating in open magnetic regions is addressed elsewhere.[32,33]

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