Muon propagation in rock, water, ice

The classical way to describe the average muon energy loss is

\[
\frac{dE_\mu}{dr} = -a - bE_\mu
\]

where

\[ h = h_{br} + h_{pp} + h_{ph} \]

accounts for the three radiation processes:
- bremsstrahlung
- production of electron positron pairs
- photoproduction

and \( a \) accounts for ionization losses.

Very roughly \( a \) is 2 MeV/(g/sq.cm) and

\( b \) is \( 4 \times 10^{-6} \) (g/sq.cm)

Using this definition one can estimate how much energy muons lose in propagation.
Using this definition of the energy loss the muon energy after propagation on $X$ g/sq.cm will be

$$E_\mu = (E_\mu^0 + \epsilon) \times \exp(-\frac{\alpha}{\beta}X) - \epsilon$$

where $\epsilon = \alpha/\beta \approx 500$ GeV.

The inverse relation is

$$E_\mu^0 = (E_\mu + \epsilon) \times \exp(\frac{\alpha}{\beta}X) - \epsilon$$

and the minimum muon energy to propagate to a depth $X$ is

$$E_{\mu}^{\text{min}} = \epsilon [\exp(\frac{\alpha}{\beta}X) - 1]$$

These expressions are important when a muon of certain energy is detected underground (or underice) and we are interested in its energy on the surface.
We know the electron bremsstrahlung, where the energy loss is

$$\frac{dE}{dx} = -\frac{N}{A} \int_0^{E-mc^2} \sigma_{br}(E,k) k \, dk$$

$$\sigma_{br} = \frac{4Z^2 \alpha \hbar^2}{k} F(E,k)$$

$$\xi \equiv 100Me^2 \frac{k}{EE-k} \frac{1}{Z^{-1/3}}.$$ is the screening parameter and

$$F(E,k) = \left[\frac{4(1-u)/3+u^2}{3} \right] \ln Z^{-1/3} + (1-u)/9 \quad \text{with } u = k/E$$

Because of the $1/k$ term the bremsstrahlung cross section becomes infinite at very low $k$, but the energy loss is still finite.

$$\frac{dE}{dx} = \frac{4NZ}{A} \alpha \hbar^2 E \left[ \ln 191 Z^{-1/3} + 1/18 \right]$$
The bremsstrahlung cross section is the base for introduction of the radiation length that gives the average amount of matter for bremsstrahlung energy loss

\[ X_0 \equiv \left[ \frac{4NZ(Z+1)}{A} \alpha r_e^2 \ln(191Z^{-1/3}) \right]^{-1} \]
Gamma ray pair production

\[
\sigma_{\gamma\gamma}(k, E) = \sigma_{\text{BR}}(E, k) \frac{E^2}{k^2} = \frac{4Z^2 \alpha r_\infty^2}{k} G(k, E)
\]

\[
\sigma_{\gamma\gamma}(k) = \int_{mc^2}^{k - mc^2} \sigma_{\text{pp}}(k, E) dE
\]

\[
= 4Z^2 \alpha r_\infty^2 \ln(191Z^{-1/3}) \frac{1}{9} - \frac{1}{54}
\]

\[
F(k, \nu) = \frac{4Z^2 \alpha r_\infty^2}{3k^2} (1 - \nu^2)
\]

\[
\nu = E/k
\]
The muon photoproduction cross section also has to deal with virtual photons that interact with nucleons.

\[ \frac{d\sigma_{\gamma \mu}(E_\mu)}{d\nu} = \frac{A\alpha}{2\pi} \sigma_{\gamma N}(vE_\mu) \nu \times \mathcal{F}[E_\mu, \nu, \sigma(vE_\mu)] \]

The gamma ray photoproduction cross section has a resonant peak character above threshold and at higher energy is roughly 1/100 of the $NN$ cross section. It is very well known from accelerator measurements.
The traditional experimental data on the muon energy loss (and muon production in the atmosphere) comes from the depth-intensity relation: the integral flux of muons as a function of the column depth. With the energy loss formulae it becomes

\[ F_\mu^{vert} = \frac{K e^{-\alpha+1}}{\alpha-1} \times \exp(-(\alpha-1)hX) \times (1-e^{-\Delta X})^{-\alpha+1} \]

The unit of depth (km.w.e.) is used always (calculated with the density of the overburdain) although the energy loss is different in different materials.
Comparison of the data taken in rock to data taken in water and ice.

One can see the increased energy loss at low energy and the decreased energy loss at high energy of the muons and water and ice.
Muon energy loss

Ionization energy loss is governed by the same formulae as the electron one after the mass, energy and velocity parameters are these for muons.

\[
\frac{dE}{dr} = -\frac{N_A Z}{A} \frac{2\pi(2e^2)^2}{Mv^2} \left[ \ln \frac{2Mv^2\gamma^2W}{I^2} - 2\gamma^2 \right]
\]

The ionization loss is logarithmically increasing with the muon energy. The 2 MeV number is the average for different materials when the muon energy is close to 1 GeV.
The differential bremsstrahlung cross section for muons is smaller than that of electrons by the mass ratio squared

\[
\frac{d\sigma_{Br}}{du} = \alpha \left( 2Zr_e \frac{m_e}{m_\mu} \right)^2 \frac{1}{u} \left[ \frac{4}{3} (1-u) + u^2 \right] \xi(\delta)
\]

and a screening factor of

\[
\xi(\delta) = \ln \left[ f_n \frac{m_\mu}{m_e} \frac{189Z^{-1/3}}{1 + (\delta/m_e)\sqrt{e189Z^{-1/3}}} \right]
\]

The pair production cross section for muons is higher than the bremsstrahlung one. Its general form is

\[
\frac{d\sigma_{p\delta r}}{dv} = \frac{2\alpha^2 r_e^2}{3\pi} Z^2 \frac{1-u}{u} \int \left[ F_e(r) + \frac{m_e^2}{m_\mu^2} F_\mu(r) \right] dr
\]

where \( v \) is the fractional energy loss of the muon – the energy of the virtual photon and the integral is over the distribution of the energy between the positron and electron.
The functions $F$ have a very complicated form.

$$F_e = \left[ \ln \left( 1 + \frac{1}{\zeta} \right) \left( (2 + r^2)(1 + \zeta) + \zeta(3 + r^2) \right) + \frac{1 - r^2 - \zeta}{1 + \zeta} - (3 + r^2) \right] L_e ,$$

\hspace{0.5cm} \text{(7.11)}

\hspace{0.5cm} \text{with}

$$\zeta = \left( \frac{v m_e}{2m_e} \right)^2 \frac{(1 - r^2)}{(1 - u)} ; \quad \zeta = \frac{v^2}{2(1 - u)}$$

\hspace{0.5cm} \text{and}

$$L_e = \ln \left( \frac{189Z^{-1/3} \sqrt{(1 + \zeta)}(1 + \nu_e)}{1 + \frac{2m_e \zeta}{E}\sqrt{1 - \zeta} Z^{-1/3}(1 + \zeta)(1 + \nu_e)} \right) - \frac{1}{2} \ln \left[ 1 + \left( \frac{3m_e}{2m_{\mu}} Z^{1/3} \right)^2 (1 + \zeta)(1 + \nu_e) \right] ,$$

\hspace{0.5cm} \text{The other $F$ term is}

$$F_\mu = \ln \left( 1 + \zeta \right) \left[ (1 + r^2)(1 + \frac{3\zeta}{2}) - \frac{1}{\zeta} (1 + 2\zeta)(1 + r^2) \right] L_\mu$$

\hspace{1cm} + \left[ \zeta(1 - r^2 - \zeta) - (1 + 2\zeta)(1 - r^2) \right] L_\mu ,$$

\hspace{0.5cm} \text{with}

$$L_\mu = \ln \left( \frac{189Z^{-2/3} \frac{2m_{\mu}}{8m_e} \sqrt{\zeta} Z^{-1/3}(1 + \zeta)(1 + \nu_\mu)}{1 + \frac{2m_e \zeta}{E} \sqrt{189Z^{-1/3}(1 + \zeta)(1 + \nu_\mu)}} \right)$$

\hspace{0.5cm} \text{and}

$$\nu_\mu = \frac{4 + r^2 + 3\zeta(1 + r^2)}{1 + (1 + r^2)(\frac{3 + \zeta}{2}) \ln (3 + \zeta) - \frac{3}{2} r^2}$$

Cross section is high but the energy loss per interaction is small because of the steep energy spectrum of the virtual photon associated with the muon.
Energy dependence of the radiation energy loss lines – standard rock (A=22, Z=11) points – ice
Total energy loss in ice. Note that ionization and pair production energy loss is almost continuous (with small fluctuations) while the bremsstrahlung and photoproduction losses are highly stochastic. Because of that the best way to study muon propagation is in Monte Carlo calculations.

The cross-over btw continuos and stochastic energy loss is at about 1 TeV
The classic definition is that only ionization loss is soft, so it could be treated continuously. The softest radiation process is pair production – high cross section and small energy loss per interaction. The figure shows what fraction of the energy loss is in hard processes in these two definitions.
There are different ways to calculate correctly the muon energy loss:
1) to use all processes as continuous with the right energy dependence.
2) Monte Carlo calculation that accounts for all fluctuations in the radiation processes.

Dependence of the muon energy on the surface on the muon energy on top of IceCube. The calculation is done using method 1). The surface energy of vertical muons reaching the top of IceCube with 100 GeV is 626 GeV. The energy ratio increases at higher energy because of the increase of energy loss.
Muon survival probability

For deep underground muon detectors the most important quantity calculated in MonteCarlo was the muon survival probability, i.e. energy dependent probability that a muon with given surface energy will survive to reach depth \( X \). The units of depth are km.w.e. In standard rock: \((A=22, Z=11, 2.65 \text{ g/ccm})\), i.e. 1 km depth = 2.65 km.w.e.

The muon energies are, left to right 1, 3.16, 10, 31.6 TeV

Arrows show depth reached without an account for the fluctuations. Different lines show bremsstrahlung cross section uncertainty.
In the case of IceCube, as well as for deep water detectors, one has to calculate not only the survival probability, but also the fluctuations of the energy loss inside the detectors.

With increasing zenith angle the pathlength increases and so does the surface energy. 626 GeV for vertical muons turn into 4 TeV for $\cos(\theta) = 0.25$. 
Average muon energy loss in different thickness of ice (or pathlength InIce) as a function of the initial muon energy. This is a Monte Carlo result. Note the irregular behavior of muons of energy less than 400 GeV – they lose energy and stop before penetrating IceCube. The shape of the curves is determined by the ratio of soft to hard energy loss.
The average energy loss is very smooth but the fluctuations on propagation are very large and difficult to account for in analysis.

Fluctuations increase with the muon energy as radiation loss starts to dominate.
Monte Carlo propagation of 1,000 muons through 1 km of ice – an attempt to establish amount of energy loss versus depth. Fluctuations are still significant.

Comparison between the energy loss of a single 2 TeV muon and three muons of total energy 2 TeV. The muon bundle has a slightly steeper decline, that can hardly be determined experimentally.