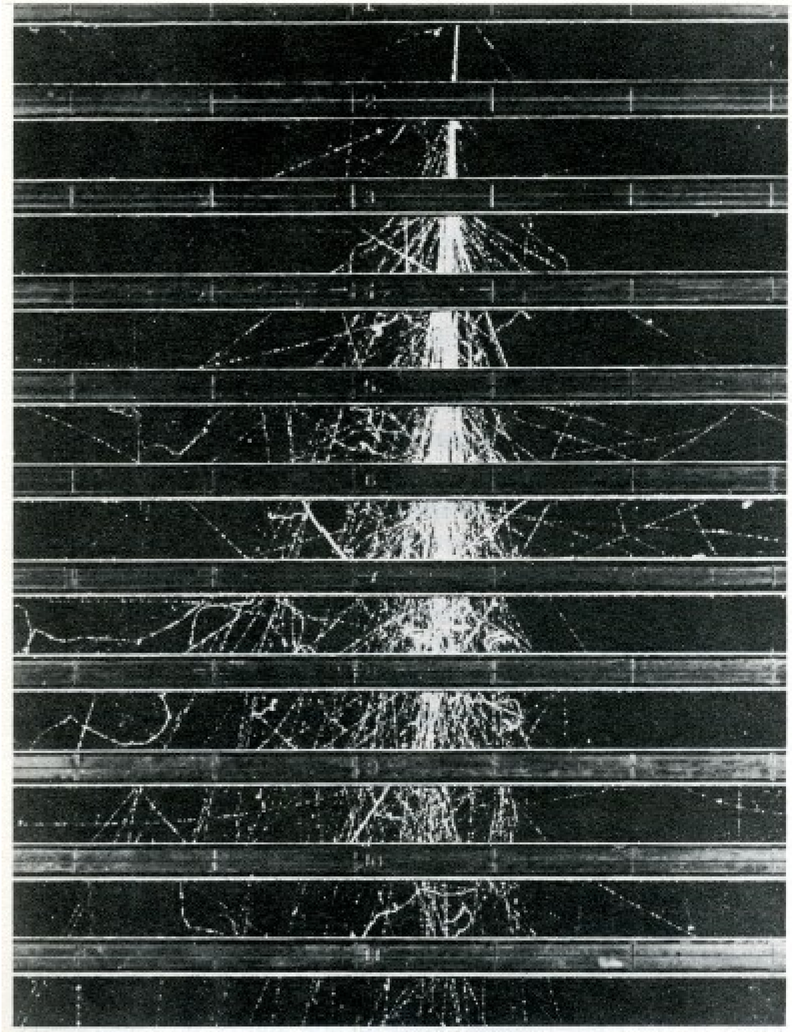
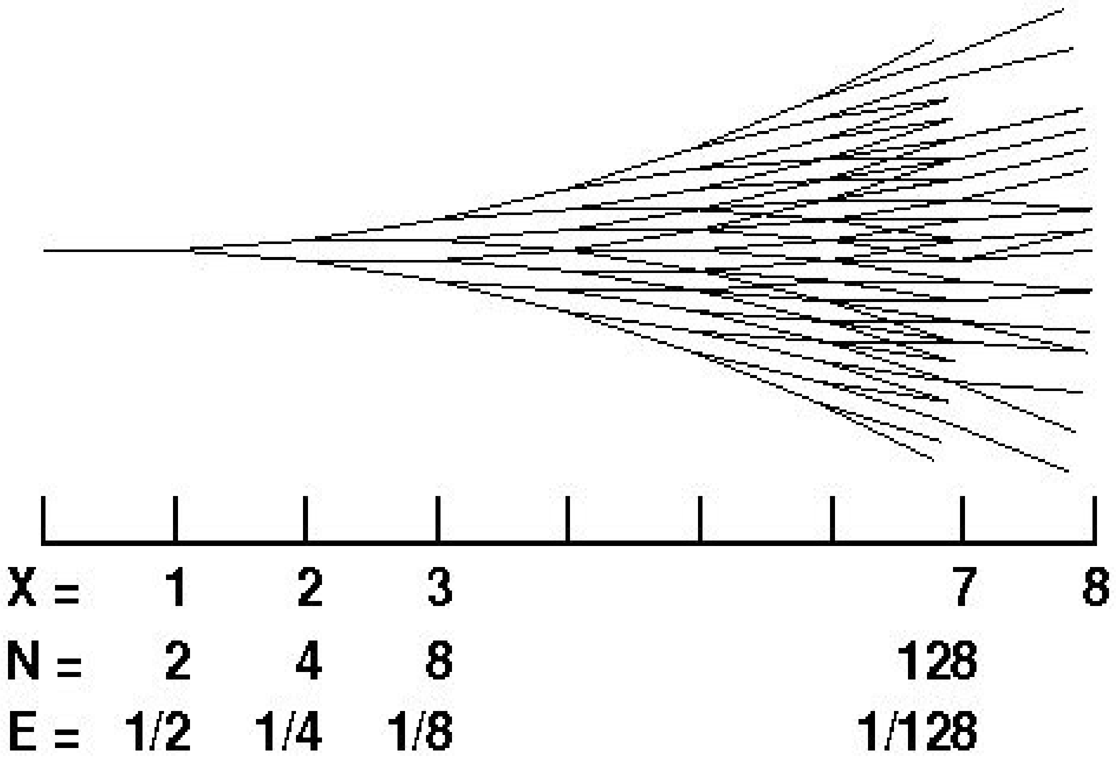
Electromagnetic showers: shower theory and Monte Carlo calculations

The shower theory was developed in 1930's when quantum electrodynamics (QED) was the most fashionable field of physics. Experimentally cascades were observed since the 1920's.

All famous physicists of that time, from Bhabha to Landau and Oppenheimer, wrote and solved cascade equations their own way. Toward the end of that period, in 1941, Heitler explained with his *toy cascade* the main features of the cascade development.



This picture is taken by the MIT Cosmic rays group in 1933. The Observations started in 1920's And continued all through the 1930's. An excellent review paper Of experimental and theoretical Developments was published by B. Rossi and K. Greisen in 1941.



There is only one type of particles in Heitler's cascade. They have fixed interaction length. Every time when these particles interact they generate two particles that share their energy. This way the number of particles increases and their energy declines. Energy conservation.

$N = 2^n$, $E = 1/2^n$, where n is # of interactions
$X_{max} = \lambda \log_2(E_0/E_c)$ particles of energy lower than E_c do not interact.

(Z bremsstrahlung)

$$e + \gamma \rightarrow e + \gamma$$

$$e^{+} + e^{-} \rightarrow \gamma + \gamma$$

$$\gamma + \gamma \rightarrow e^{+} + e^{-}$$

(Z pair production)

Bremsstrahlung – full screaning

$$\left.\frac{d\sigma}{d\varepsilon}\right|_{e\rightarrow e+\gamma}(v;E) = 4\ Z^2\ \alpha\ r0^2$$

$$\frac{1}{v}\ \left\{\left[1+(1-v^2)-\frac{2}{3}(1-v)\right]\ \ln\left(183\,Z^{-1/3}\right)+\frac{1}{9}(1-v)\right\}$$

Pair production – full screaning

$$\left.\frac{d\sigma}{du}\right|_{\gamma\rightarrow e^+e^-}(u;K) = 4\ Z^2\ \alpha\ r0^2$$

$$\left\{\ \left[u^2+(1-u)^2+\frac{2}{3}u\,(1-v)\right]\ \ln\left(183\,Z^{-1/3}\right)-\frac{1}{9}u(1-u)\right\}$$

Radiation length

$$\lambda_{\mathrm{rad}} = 4\,\alpha\, r_0^2\, \frac{N_A}{A}\, Z^2\, \ln[183\, Z^{-1/3}]$$

The length in a material on which the electron energy is decreased to E/e. The pair production interaction length for photons is 9/7 radiation lengths, i.e. gamma rays interact not as often as electrons do.

The development of the number of gamma rays of energy E in the cascade is described by the following equation:

loss to pair production

$$\frac{\partial n_\gamma}{\partial t}(E,t) = -n_\gamma(E,t) \int_0^1 du\, \psi(u)$$

$$+ \int_E^\infty dE' \int_0^1 dv\, n_e(E',t)\, \varphi(v)\, \delta[E - v\,E']$$

gain from bremsstrahlung

The equation for electrons is more complicated because the electron does not disappear after bremsstrahlung, it survives with lower energy. The three terms in the equation represent 1) loss because of decreased electron energy; 2) gain fron decreased electron energy; 3) gain from pair production.

$$\frac{\partial n_e}{\partial t}(E,t) = -n_e(E,t) \int_0^1 dv\, \varphi(v)$$

$$+ \int_E^\infty dE' \int_0^1 dv\, n_e(E',t)\, \varphi(v)\, \delta[E-(1-v)\,E']$$

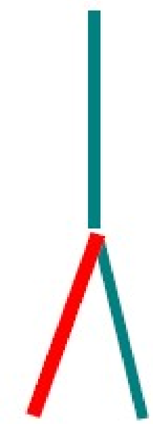
$$+ \int_E^\infty dE' \int_0^1 du\, n_\gamma(E',t)\, \psi(u)\, \delta[E-u\,E']$$

The energy of the primary particle is always shared by two electromagnetic particles.

bremsstrahlung

$$\varphi(v) = \left[\frac{d\sigma}{dv}(v)\right]_{\mathrm{brems}} \left(\frac{N_A}{A}\;\lambda_{\mathrm{rad}}\right)$$

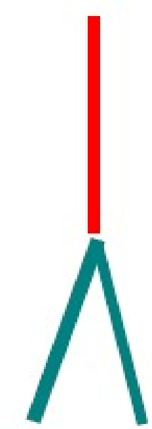
$$\varphi(v) = \frac{1}{v}\;\left[1 - \left(\frac{2}{3} - 2\,b\right)\;(1-v) + (1-v)^2\right]$$



pair production

$$\psi(u) = \left[\frac{d\sigma}{du}(u)\right]_{\mathrm{pair}} \left(\frac{N_A}{A}\;\lambda_{\mathrm{rad}}\right)$$

$$\psi(u) = (1-u)^2 + \left(\frac{2}{3} - 2\,b\right)\;(1-u)\;u + u^2$$

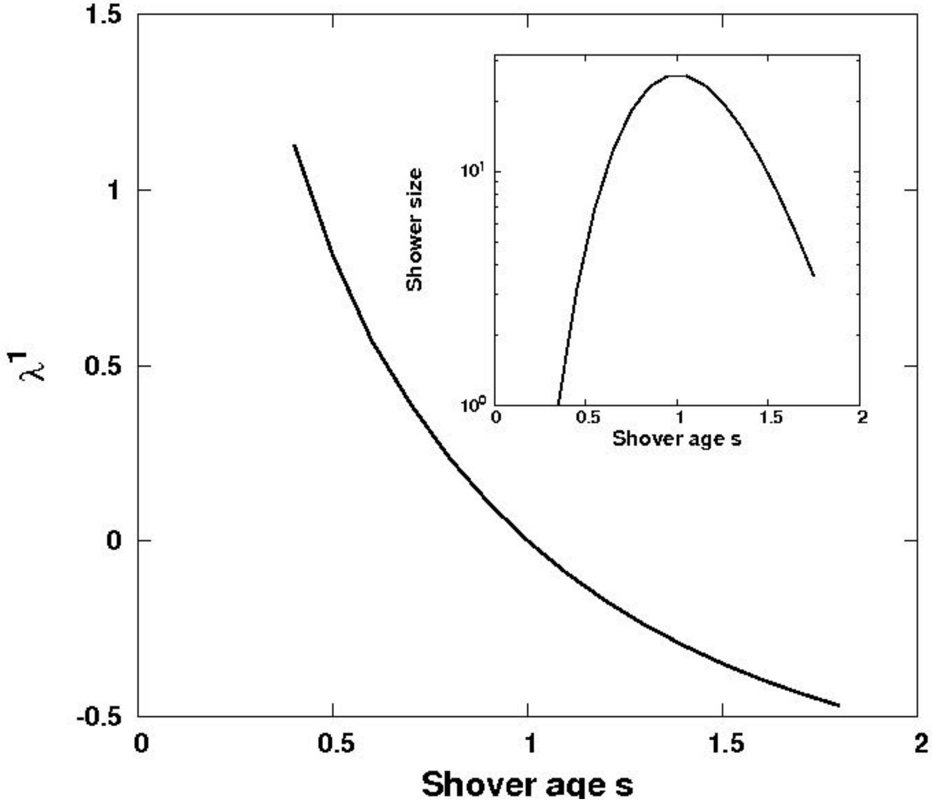


The solution of the cascade equations by Rossi & Greisen gives the number of particles at depth *X* (radiation lengths) from a primary particle of type *j* and energy E_0.

$$\pi^j, \gamma^j = \frac{\mathcal{H}_1^j}{\sqrt{2\pi}\, s^n (\lambda_1'' X + n/s^2)^{1/2}} \left(\frac{E_0}{E}\right)^s \frac{1}{E} e^{\lambda_1 X}$$

The shower age parameter *s* describes the stage of the shower development which is related to the depth *X* in radiation lengths as

$$X = -\frac{1}{\lambda_1'(s)} \left[\beta - \frac{n}{s}\right]$$



Shower development depends on the value of the root $\lambda_1(s)$ in the $\exp(X.\lambda_1(s))$

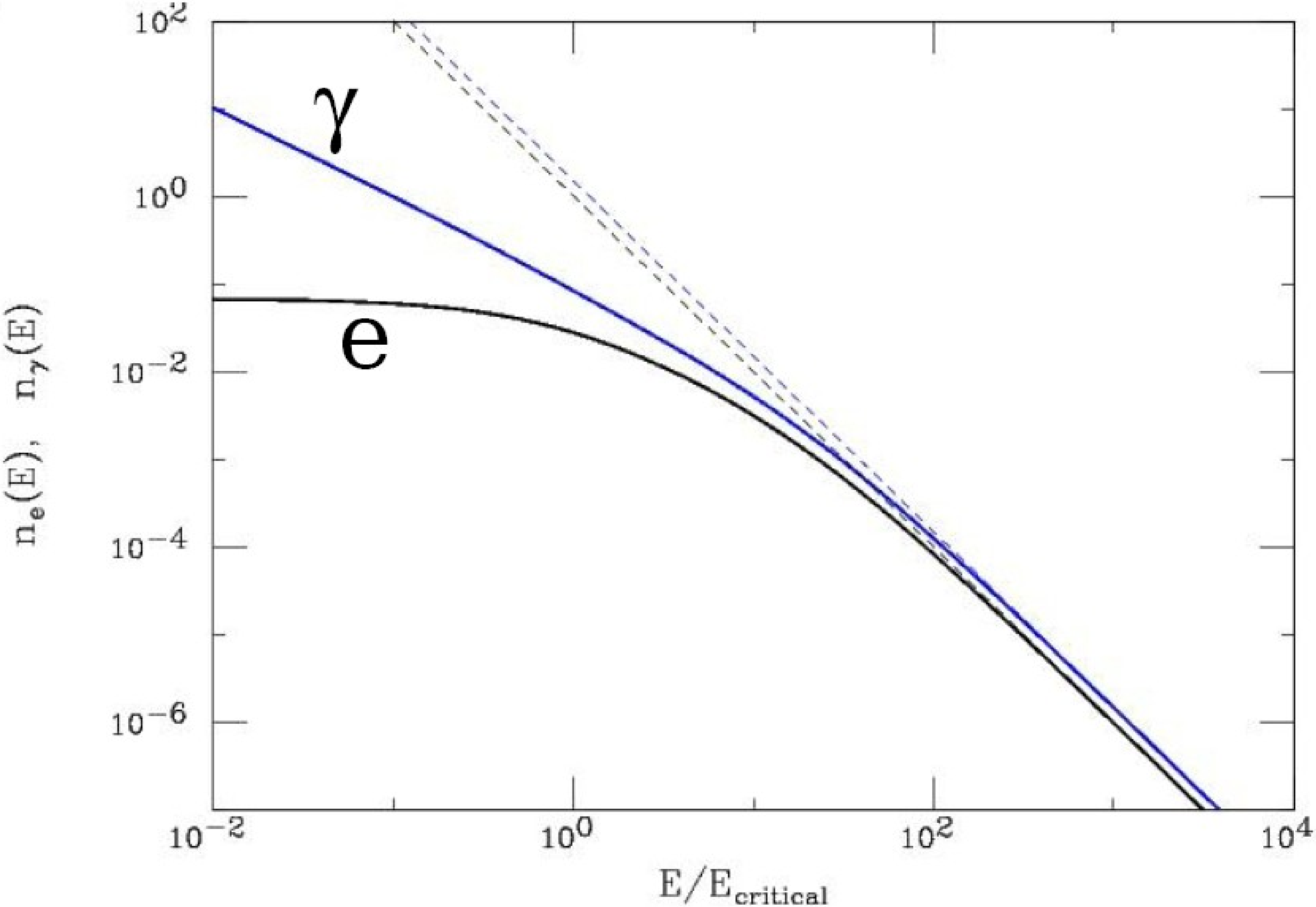
There are two approximate solution of the cascade equations:

Appr. A, no ionization energy loss, i.e. only E/E_0 matters, no energy scale

Appr. B, ionization loss accounted for.

Only high energy constant bremsstrahlung and pair production cross sections are used in both.

Differences between Appr. A and Appr. B



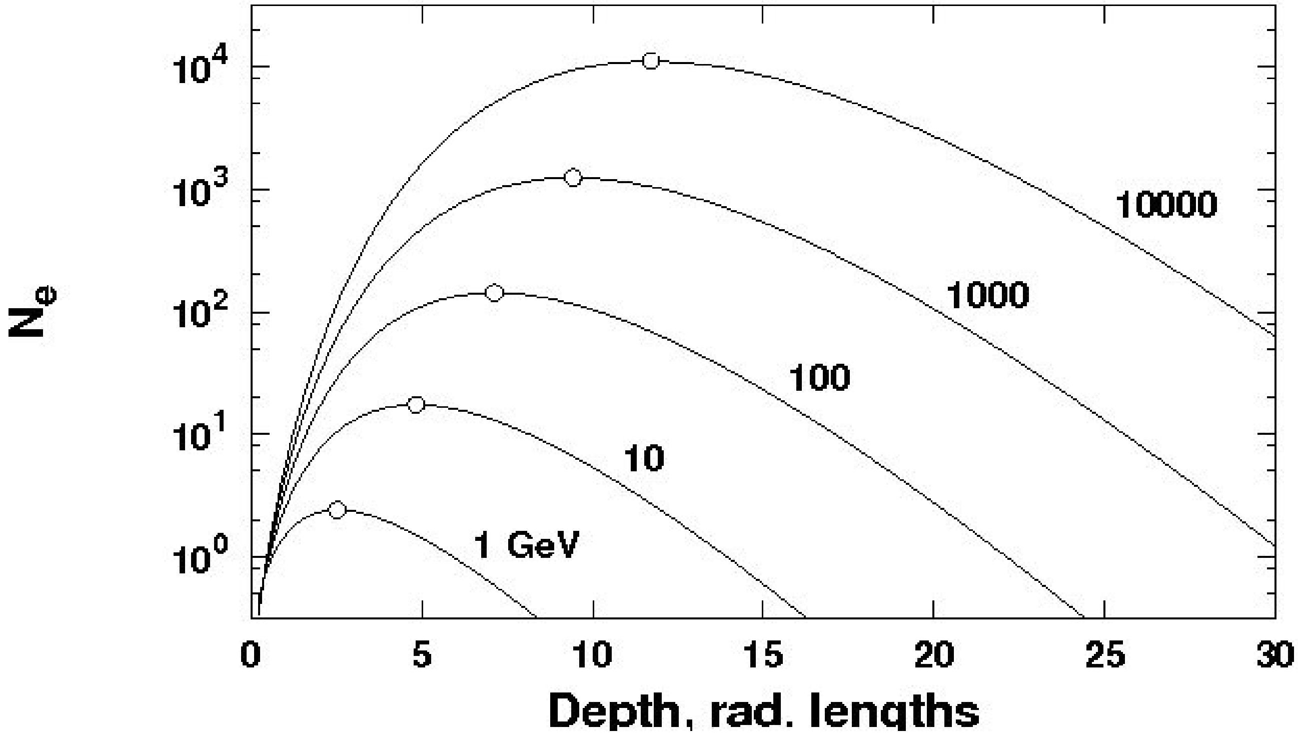
$$\lambda_1(s) \approx (s - 1 - 3\ln s)/2\,; \quad \lambda_1'(s) \approx \frac{-\beta}{X}\,; \quad \lambda_1'' \approx \frac{3}{2s^2}$$

$$s = \frac{3X + 2n}{X + 2\beta}$$

The equations above are differential in particle energy. To obtain the integral number of particles above energy E one has to integrate in energy.

$$\mathcal{P}^i(E_0, E, X) = \int_E^{E_0} p^i(E_0, E', X)\, dE'$$

In Appr. B the energy E is replaced by the critical energy ε where the ionization energy loss equals the radiation energy loss, close to 80 MeV in air and in water.

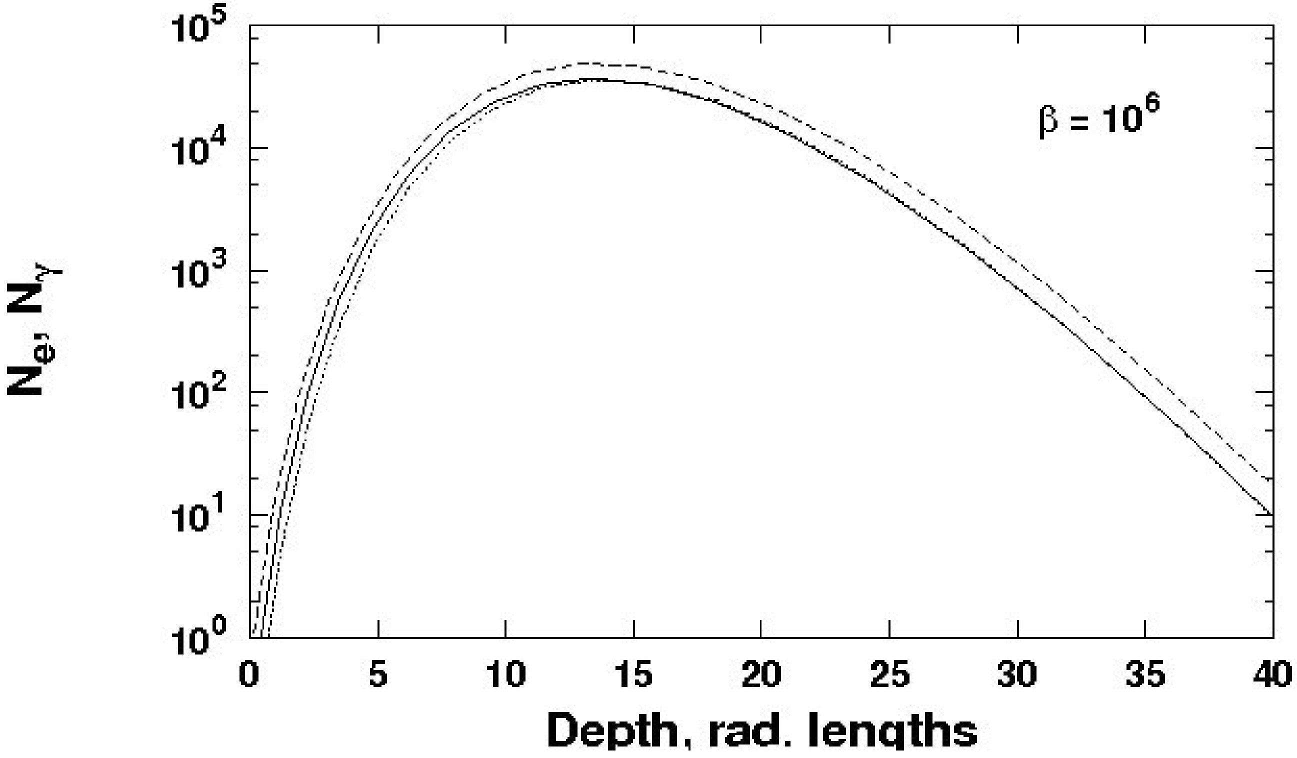


Greisen's fits are

$$N_e^\gamma = \frac{0.135}{\sqrt{\beta}} \exp\left[X\left(1 - \frac{3}{2}\ln s\right)\right] \text{ for Appr. A}$$

$$N_e^\gamma = \frac{0.31}{\sqrt{\beta_0}} \exp\left[X\left(1 - \frac{3}{2}\ln s\right)\right] \text{ for Appr. B}$$

$$s = \frac{3X}{X + 2\beta}$$



Comparison of the exact solutions of Appr. A with Greisen's parametrization (dots). It is very good around shower maximum.

Angular and lateral distribution of the shower particles

$$\langle \delta\theta^2 \rangle = \left(\frac{E_s}{E}\right)^2 \delta X \qquad\qquad E_s = m_e c^2 \sqrt{4\pi/\alpha}$$

The Moliere length $r_1 = \left(\frac{E_s}{E_c}\right) X$ is ¼ of the radiation length since E_s is 21 MeV. The solution for the lateral distribution of electrons, the NKG function is

$$\rho_e(r, X) = N_e(X) \frac{C(s)}{r r_1} \left(\frac{r}{r_1}\right)^{s-1} \left(1 + \frac{r}{r_1}\right)^{s-9/2}.$$

where *C(s)* is a normalization coefficient derived from the requirement

$$\frac{2\pi}{N_e(X)} \int_0^\infty r\,\rho(r)\,dr = 1$$

Monte Carlo calculations of electromagnetic cascades

Advantages:

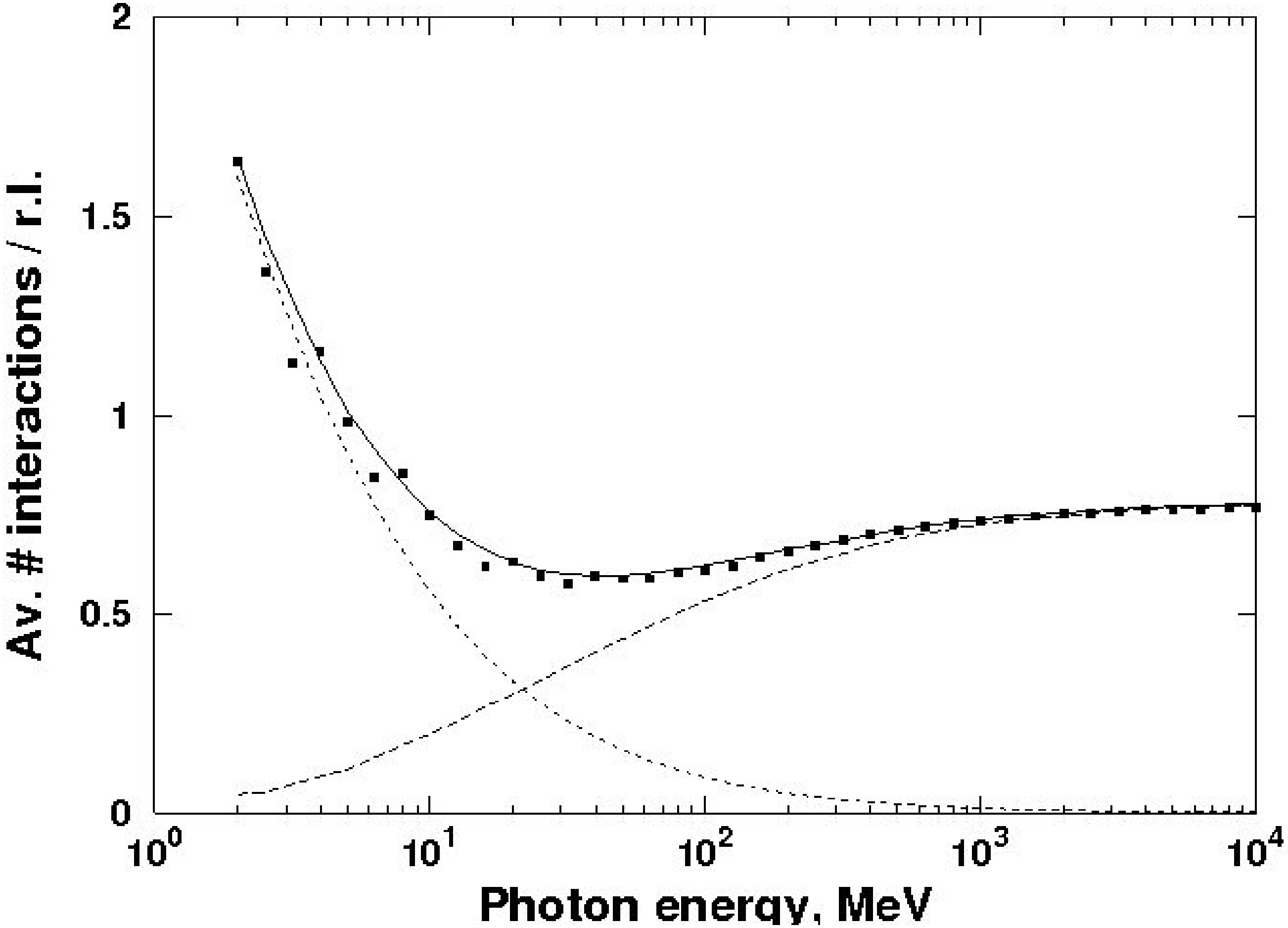
1) use the exact cross sections for bremsstrahlung and pair production

2) use the correct energy dependence of ionization loss

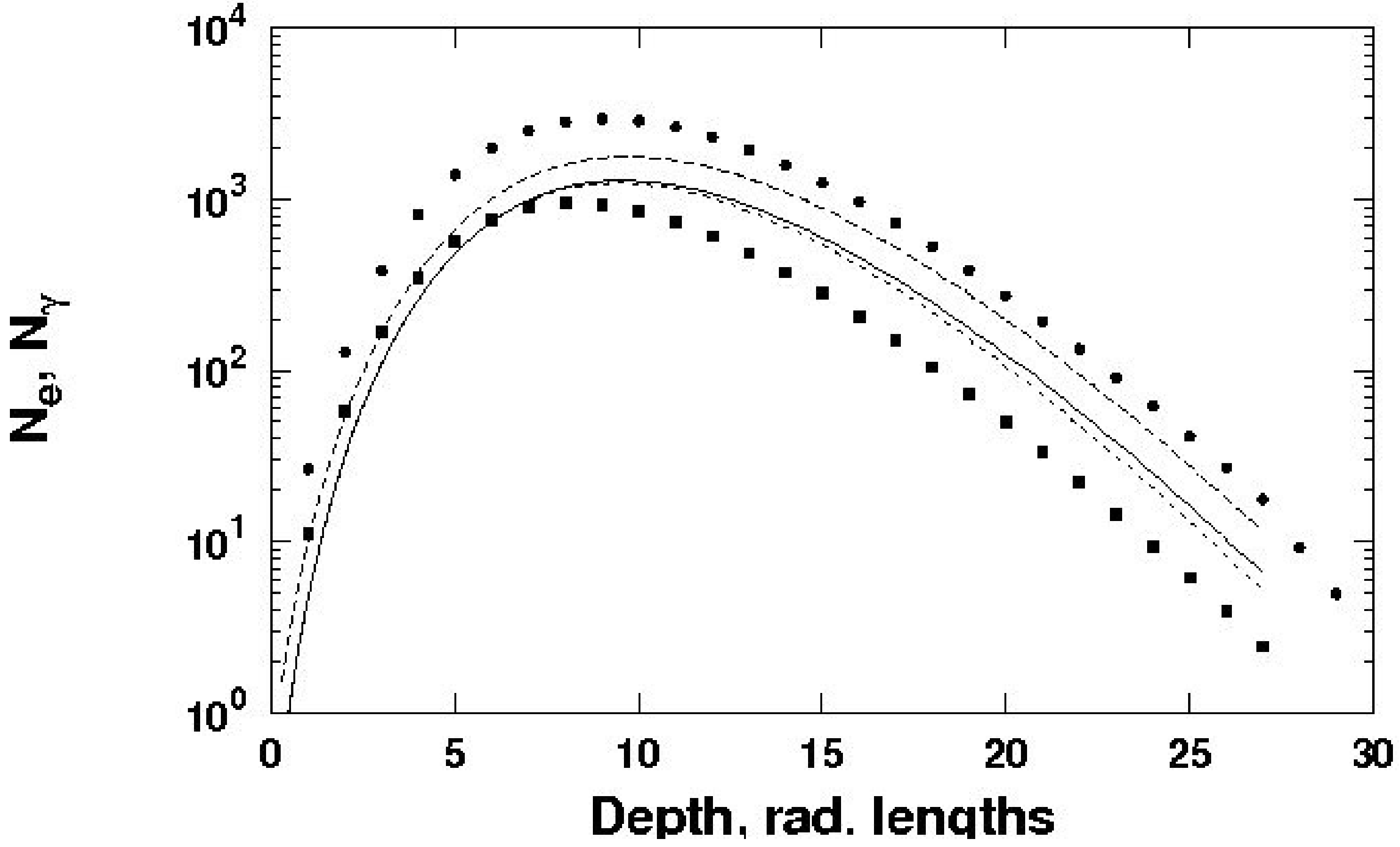
3) may include all electromagnetic interactions, Compton scattering very important for gamma rays below 20 MeV it exceeds the pair production cross section.

Give correct account for the fluctuations in the shower development, as well as for the angular and lateral distribution of the shower particles.

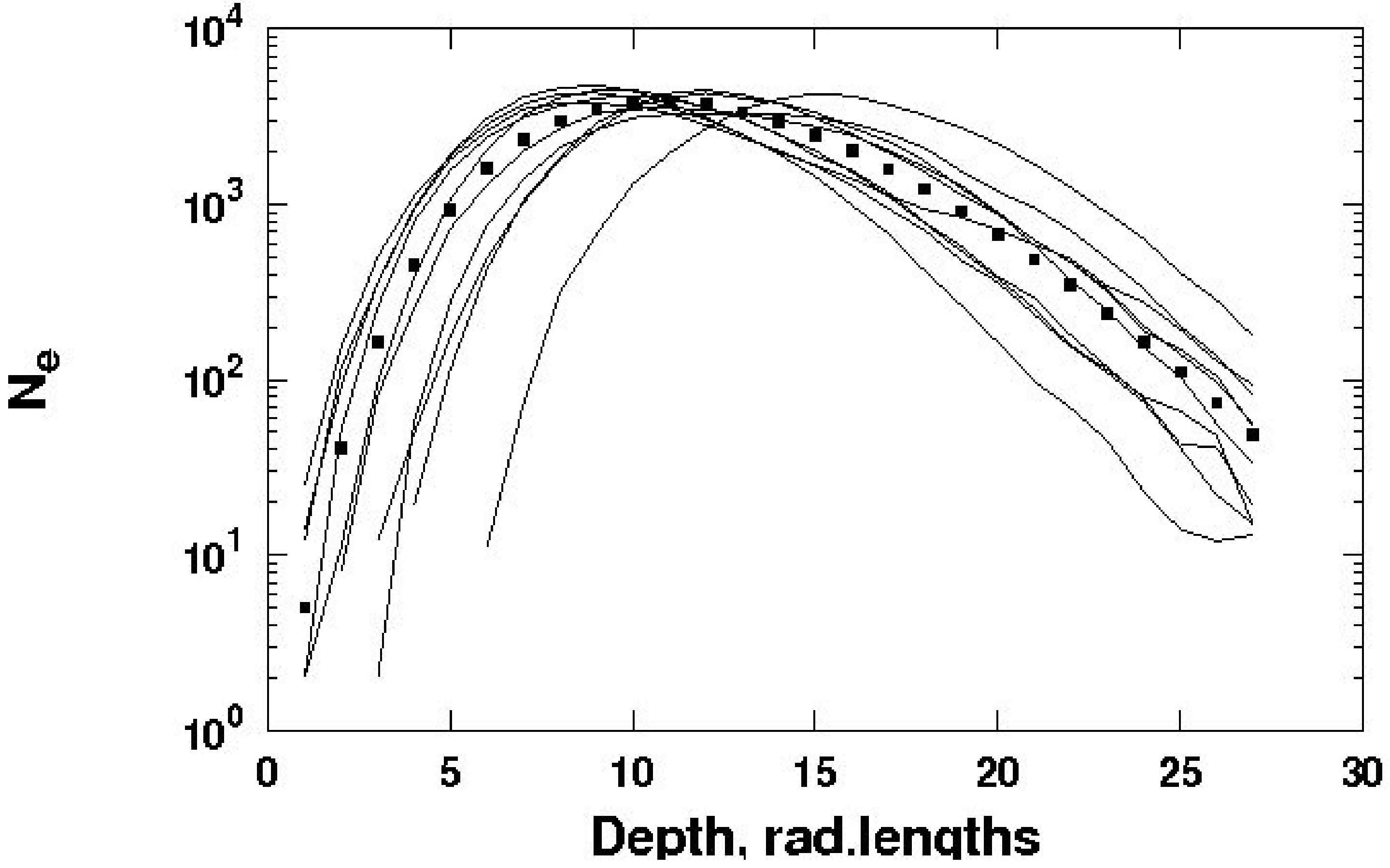
Very good methods developed in late 1950's and early 1960's. Best current code is EGS4 coming out of the work by Ford & Nelson. It implements H. Messel's scheme of modeling simultaneously secondary particle energy and the energy dependent cross section



Comparison between Appr. B and a Monte Carlo calculation with energy threshold of 10 MeV. The dotted line is Greisen's formula.



Monte Carlo calculations make a good sense in the estimate of the fluctuations in the shower development. See below the shower profiles of 10 individual showers and their average number of electrons.



Lateral distribution of photons and electrons of energy above 1.5, 10, and 100 MeV in a 1 TeV photon initiated shower. Guess what is what. Solid curve is the NKG formula and the dashed curve is a fit.