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Part II: Stellar Structure and Evolution

We have seen in part I how a star’s color or peak wavelength $\lambda_{\text{max}}$ indicates its characteristic temperature near the stellar surface. But what about the temperature in the star’s deep interior? Intuitively, we expect this to be much higher than at the surface, but under what conditions does it become hot enough to allow for nuclear fusion to power the star’s luminosity? And how does it scale quantitatively with the overall stellar properties, like mass $M$, radius $R$, and perhaps luminosity $L$?

To answer these questions, let us identify two distinct considerations for our intuition that the interior temperature should be much higher than at the surface.

The first we might characterize as the “blanketing” by the overlying layers, which traps any energy generated in the interior, much as a blanket in bed traps our body heat, keeping our skin temperature at a comfortable warmth, instead of the relative chilliness of having it exposed to open air. In this picture, the equilibrium interior temperature depends on the rate of energy generation (from metabolism for our bodies, or nuclear fusion for stars) and the ‘insulation thickness’ of the overlying of material to the surface (given by the optical depth; see §14.)

But distinct from this consideration of the transport of energy from the interior, there is for a star a dynamical requirement for force or momentum balance, to keep the star supported against the inward pull of its own self-gravity. Since stars are gaseous, without the tensile strength of a solid body, this gravitational support is supplied by increased internal gas pressure $P$, allowing the star to remain in a static equilibrium. This high gas pressure arises from a combination of high density and high temperature. As detailed in §13.3, this allows us to determine a characteristic interior temperature, through a further application of the Virial theorem for bound systems that was briefly discussed for bound orbits in part I.

13. Balance between Pressure and Gravity

13.1. Hydrostatic equilibrium

To quantify this gravitational equilibrium for a static star, consider, as illustrated in figure 13.1, a thin radial segment of thickness $dr$ with local density $\rho$ and downward gravitational acceleration $g$. The mass-per-unit-area of this layer is $dm = \rho dr$, with corresponding weight-per-unit-area $g dm$. To support this weight, the gas pressure at the lower end of this
layer must be higher by amount $|dP| = gd m = \rho g dr$ than the upper end, implying

$$\frac{dP}{dr} = -\rho g,$$

a condition know as **Hydrostatic Equilibrium**.

For an **ideal gas**, the pressure depends on the product of the number density $n = \rho/\mu$ and temperature $T$,

$$P = nkT = \rho \frac{kT}{\mu} \equiv \rho c_s^2,$$

where $k = 1.38 \times 10^{-16}$ erg/K is Boltzmann’s constant, $\mu$ is the average mass – a.k.a. the “mean molecular weight” – of all particles (i.e., both ions and electrons) in the gas, and the final equation defines the isothermal\(^1\) sound speed, $c_s \equiv \sqrt{kT/\mu}$.

For any given element, the **fully ionized** molecular weight is just set by the nuclear mass $Am_p$ (because the electron mass is by comparison negligible) divided by the nuclear charge number plus one (for the one nucleus + $Z_n$ electrons that balance the nuclear charge $eZ_n$), $\mu = m_p A/(Z_n + 1)$. For a gas mixture with mass fraction $X$, $Y$, and $Z$ for H, He, and metals,

\(^1\)This speed, which was first derived by Newton, would only be the speed of sound if the gas remained strictly constant temperature (isothermal). In practice, the temperature fluctuations associated with the gas compressions make the actual “adiabatic” speed of sound slightly higher, by a factor $\sqrt{\gamma}$, where $\gamma$ is the ratio of specific heats ($5/3$ for a monatomic gas).
the overall mean molecular weight is then obtained by a weighted average of the inverses \(m_p/\mu = (Z_n + 1)/A\) of the individual components (as in a parallel circuit), yielding

\[
\bar{\mu} = \frac{m_p}{2X + 3Y/4 + Z/2} \approx 0.6m_p \equiv \bar{\mu}_\odot,
\]

where the last equality is for the solar case with \(X = 0.72\), \(Y = 0.26\), and \(Z = 0.02\). More generally, for fully ionized gases the proton-mass-scaled molecular weight \(\bar{\mu}/m_p\) can range from 1/2 for pure H (\(X = 1\)), to 4/3 for pure He (\(Y = 1\)), to a maximum of 2 for pure heavy metals (\(Z = 1\)).

### 13.2. Pressure scale height and thinness of surface layer

The ratio of eqns. (13.2) to (13.1) defines a characteristic pressure scale height,

\[
H \equiv \frac{P}{|dP/dr|} = \frac{kT}{\bar{\mu}g} = \frac{c_s^2}{g},
\]

where the absolute value of the pressure gradient \(dP/dr\) (which itself is negative) ensures the scale height is positive.

At the stellar surface radius \(r = R\), where the gravity and temperature approach their fixed surface values \(g_* = GM/R^2\) and \(T = T_*\), the scale height becomes quite small, typically only a tiny fraction of the stellar radius,

\[
\frac{H}{R} = \frac{kT_*/\bar{\mu}}{GM/R} = \frac{2c_{ss}^2}{V_{esc}^2} \approx 0.0005 \frac{T_*/T_\odot}{\bar{\mu}/\bar{\mu}_\odot} \frac{R/R_\odot}{M/M_\odot},
\]

where \(V_{esc}\) is the escape speed introduced in part I. For the solar atmosphere, the sound speed is \(c_{ss} \approx 9\) km/s, about 1/60th of the surface escape speed \(V_{esc} = 620\) km/s.

If we further idealize a stellar atmosphere as being roughly isothermal, i.e. with a nearly constant temperature \(T \approx T_*\), then, since the gravity is also effectively fixed at the surface value, we see that the scale height also becomes constant. This makes it easy to integrate the hydrostatic equilibrium equation (13.1), thus giving the variation of density and pressure in terms of a simple exponential stratification with height \(z \equiv r - R\),

\[
\frac{P(z)}{P_*} = \frac{\rho(z)}{\rho_*} = e^{-z/H},
\]

where the asterisk subscripts denote values at some some surface layer where \(z \equiv 0\) (or \(r = R\)). In practice the temperature variations in an atmosphere are gradual enough that quite generally both pressure and density very nearly follow such an exponential stratification.
The results in this subsection actually apply to any gravitationally bound atmosphere, not only for stars but also for planets, including the earth, with similarly small characteristic values for the ratio $H/R$. This is the basic reason that the earth’s atmosphere is confined to such a narrow layer around its solid surface, meaning that at just a couple hundred kilometers altitude it is nearly a vacuum, so tenuous that it is imparts only a weak drag on orbiting satellites.

For stars or gaseous giant planets without a solid surface, it means that the dense, opaque regions have only a similarly narrow transition to the fully transparent upper layers, thus giving them a similarly sharp visual edge as a solid body. For stars it means that models of the escape of interior radiation through this narrow atmospheric layer can essentially ignore the stellar radius, allowing the emergent spectrum to be well described by a planar atmospheric model fixed by just two parameters – surface temperature and gravity– and not dependent on the actual stellar radius.

13.3. Hydrostatic balance in stellar interior and the virial temperature

This hydrostatic balance must also apply in the stellar interior, but now both the temperature and gravity have a strong spatial variation. At any given interior radius $r$, the local gravitational acceleration depends only on the mass within that radius,

$$M(r) \equiv 4\pi \int_0^r \rho(r')r'^2 dr'. \quad (13.7)$$

This thus requires the hydrostatic equilibrium equation to be written in the somewhat more general form,

$$\frac{dP}{dr} = -\rho(r) \frac{GM(r)}{r^2}. \quad (13.8)$$

This represents one of the key equations for stellar structure.

The implications of hydrostatic equilibrium for the hot interior of stars are quite different from the steep exponential pressure drop near the surface; indeed they allow us now to derive a remarkably simple scaling relation for a characteristic interior temperature $T_{\text{int}}$.

For this consider the associated interior pressure $P_{\text{int}}$ at the center of the star ($r = 0$); to drop from this high central pressure to the near-zero pressure at the surface, the pressure gradient averaged over the whole star must be $|dP/dr| \approx P_{\text{int}}/R$. We can similarly characterize the gravitational attraction in terms of the surface gravity $g_s = GM/R^2$ times an interior density that scales as $\rho_{\text{int}} \sim P_{\text{int}}\mu/kT_{\text{int}}$. Applying these in the basic definition
of scale height (13.4), we find that for the interior \( H \approx R \), which in turn implies for this characteristic stellar interior temperature,

\[
T_{\text{int}} \approx \frac{GM\bar{\mu}}{kR} \approx 13 \times 10^6 \text{K} \frac{M/M_\odot}{R/R_\odot}.
\]  \hspace{1cm} (13.9)

Thus, while surface temperatures of stars are typically a few thousand Kelvin, we see that their interior temperatures are typically of order 10 million Kelvin! As discussed below (§16), this is indeed near the temperature needed for nuclear fusion of Hydrogen into Helium in the stellar core.

This close connection between thermal energy of the interior (\( \sim kT \)) to the star’s gravitational binding energy (\( \sim GM\bar{\mu}/R \)) is really just another example of the Virial theorem for gravitationally bound systems, as discussed in part I for the case of bound orbits. The temperature is effectively a measure of the average kinetic energy associated with the random thermal motion of the particles in the gas. Thermal energy is thus just a specific form of kinetic energy, and the Virial theorem tells us that the average kinetic energy in a bound system equals one-half the magnitude of the gravitational binding energy.

\[13.4 \text{. Questions and Exercises}\]

**Quick Question 1:**
(a.) For a typical temperature on a spring day (\( \sim 50^\circ\)F), compute the scale height \( H \) for the earth (in km), and its ratio to earth’s radius, \( H/R_e \).
(b.) Relative to values at sea level, compute the pressure and density at a typical height \( h = 300 \text{ km} \) for an orbiting satellite.

**Quick Question 2:**
(a.) Compute the escape speed (in km/s) from stars with \( M = M_\odot \) and \( R = 2R_\odot \); with \( M = 2M_\odot \) and \( R = R_\odot \).
(b.) For these stars, estimate the associated central temperature.
14. Transport of Radiation from Interior to Surface

14.1. Random walk of photon diffusion from stellar core to surface

Let us next turn to the “blanketing” effect of the star’s material in trapping the heat and radiation of the interior. Within a star the absorption of light by stellar material is counteracted by thermal emission. As illustrated in figure 14.1, radiation generated in the deep interior of a star undergoes a diffusion between multiple encounters with the stellar material before it can escape freely into space from the stellar surface. The number of mean-free-paths $\ell$ from the center at $r = 0$ to the surface at radius $r = R$ now defines the central optical depth

$$\tau_c = \int_0^R \frac{dr'}{\ell} = \int_0^R \kappa \rho dr'. \quad (14.1)$$

As discussed in Appendix D, the opacity in stellar interiors typically has a CGS value of $\kappa \approx 1 \text{ cm}^2/\text{g}$, with a minimum set by the value for Thomson scattering by free electrons, $\kappa_e \approx 0.34 \text{ cm}^2/\text{g}$. We can then estimate a typical value of this central optical depth by simply taking the density to be roughly characterized by its volume average $\bar{\rho} = M/(4\pi R^3/3)$, where again $M$ and $R$ are the stellar mass and radius. For the Sun this works out to give $\bar{\rho}_\odot \approx 1.4 \text{ g/cm}^3$, i.e. just above the density of water. (The Sun wouldn’t quite float in your bathtub.) Since the opacity is also near unity in CGS units, the average mean-free-path for scattering in the Sun is just $\bar{\ell}_\odot \approx 0.7 \text{ cm}$.

In the core of the actual Sun, the density is typically a hundred times higher than this mean value, so the core mean-free-path is a factor hundred smaller, i.e $\ell_{\text{core}} \approx 0.07 \text{ mm}$! But either way, the mean-free-path is much, much smaller than the solar radius $R_\odot \approx 700,000 \text{ km} \approx 7 \times 10^{10} \text{ cm}$. This implies the optical depth from the center to surface is truly enormous, with a typical value

$$\tau_c \approx \frac{R_\odot}{\bar{\ell}_\odot} \approx 10^{11}. \quad (14.2)$$

The total number of scatterings needed to diffuse from the center to the surface can then be estimated from a basic “random walk” argument. The simple 1D version states that after $N$ left/right random steps of unit length, the root-mean-square (rms) distance from the origin is $\sqrt{N}$. For the 3D case of stellar diffusion, this rms number of unit steps can be roughly associated with the total number of mean-free-paths between the core and surface, i.e. $\sqrt{N} \approx \tau$. This implies that photons created in the core of the Sun need to scatter a total of $N \approx \tau^2 \approx 10^{22}$ times to reach the surface!

In traveling from the Sun’s center to its surface, the net distance is just the Sun’s radius $R_\odot$; but the cumulative path length traveled is much longer, $\ell_{\text{tot}} \approx N\bar{\ell}_\odot \approx \tau^2 \bar{\ell}_\odot \approx \tau R_\odot$. For
photons traveling at the speed of light $c = 3 \times 10^{10}$ cm/s, the total time for photons to diffuse from the center to the surface is thus

$$t_{\text{diff}} = \frac{\tau}{c} \frac{L}{\ell} \approx \frac{\tau}{c} R \approx 10^{11} \times 2.3 \text{ s} \approx 7000 \text{ yr},$$

(14.3)

where for the last evaluation, it is handy to recall again that $1 \text{ yr} \approx \pi \times 10^7$ s.

Once the photons reach the surface, they can escape the star and travel unimpeded through space, taking, for example, only a modest time $t_{\text{earth}} = \frac{\text{au}}{c} \approx 8 \text{ min}$ to cross the 1 au ($\approx 215R_\odot$) distance from the sun to the earth.

A stellar atmospheric surface thus marks a quite distinct boundary between the interior and free space. From deep within the interior, the stellar radiation field would appear nearly isotropic (same in all directions), with only a small asymmetry (of order $1/\tau$) between upward and downward photons. But near the surface, this radiation becomes distinctly anisotropic, emerging upward from the surface below, but with no radiation coming downward from empty space above.
14.2. Diffusion approximation at depth

This picture of photons undergoing a random walk through the stellar interior can be formalized in terms of a diffusion model for radiation transport in the interior. Appendix C discusses the transition from diffusion to free-streaming that occurs in the narrow region near the stellar surface, known as the “stellar atmosphere”. This is described by the equation of radiative transfer, given by eqn. (14.1), with eqn. (C2) now defining the vertical optical depth \( \tau(r) \) from a given radius \( r \) to an external observer at \( r \to \infty \).

But in the deep interior layers within a star, i.e. with large optical depths \( \tau \gg 1 \), the trapping of the radiation makes the intensity \( I \) nearly isotropic and near the local Planck function \( B \). Applying this to the derivative term in eqn. (14.1) and solving for \( I \) gives an “diffusion approximation” form for the intensity,

\[
I(\mu, \tau) \approx B(\tau) + \mu \frac{dB}{d\tau},
\]

(14.4)

where we recall from §12.1 that \( \mu \) is the cosine of the angle between the ray and the vertical direction, so that \( \mu = +1 \) is directly upward, and \( \mu = -1 \) is directly downward.

Since \( dB/d\tau \) is of order \( B/\tau \), we can see that the second term is much smaller, by a factor \( \sim 1/\tau \ll 1 \), than the first, leading-order term. Recall that both the specific intensity \( I \) and the Planck function \( B \) have the same units as a surface brightness, i.e. energy/area/time/solid angle.

The local net upward flux \( F \) (energy/area/time) is computed by weighting the intensity by the direction cosine \( \mu \) and then integrating over solid angle,

\[
F \equiv \oint I\mu d\Omega = 2\pi \int_{-1}^{1} \left( \mu B(\tau) + \mu^{2} \frac{dB}{d\tau} \right) d\mu.
\]

(14.5)

Since the leading order term with \( B(\tau) \) is then odd over the range \(-1 < \mu < 1\), it vanishes upon integration, giving

\[
F \approx \frac{4\pi}{3} \frac{dB}{d\tau}.
\]

(14.6)

Again recalling that \( B = \sigma_{sb} T^{4}/\pi \), and noting the optical depth changes with radius \( r \) as \( d\tau = -\kappa \rho dr \), we can alternatively write the flux as a function of the local temperature gradient,

\[
F(r) = -\left[ \frac{4\pi}{3\kappa \rho} \frac{\partial B}{\partial T} \right] dT \frac{dT}{dr} = -\left[ \frac{16\sigma_{sb}}{3\kappa \rho} T^{3} \right] dT \frac{dT}{dr}.
\]

(14.7)

The terms in square bracket can be thought of as a radiative conductivity, which we note increases with the cube of the temperature \( T^{3} \), but depends inversely on opacity and density, \( 1/\kappa \rho \).
14.3. Atmospheric variation of temperature with optical depth

A star’s luminosity $L$ is generated in a very hot, dense central core. Outside this core, at any stellar envelope radius $r$, the local radiative flux scales as $F = L/4\pi r^2$, which near the stellar surface $r \lesssim R$ approaches the fixed surface value $F_* = L/4\pi R^2 \equiv \sigma_{sb} T_{\text{eff}}^4$, where the last equation recalls the definition of the stellar effective temperature $T_{\text{eff}}$. Since in such a surface layer $F_*$ is independent of $\tau$, eqn. (14.6) can be trivially integrated in this layer to give,

$$\frac{4\pi}{3} B(\tau) = F_* \tau + C = \sigma_{sb} T_{\text{eff}}^4 \tau + C,$$

where $C$ is an integration constant. Recalling also from eqn. (5.1) that $\pi B = \sigma_{sb} T^4$, we can convert (14.8) into an explicit expression for the variation of temperature with optical depth,

$$T^4(\tau) = \frac{3}{4} T_{\text{eff}}^4 \left[ \tau + 2/3 \right],$$

wherein, in light of the result in §C.2, we have taken the integration constant such that $T(\tau = 2/3) = T_{\text{eff}}$.

Together with the equation (13.1) for hydrostatic equilibrium, equation (14.7) for radiative diffusion determines the fundamental structure of the stellar interior. Let us next apply these to explain the empirical mass-luminosity relation $L \sim M^3$.

14.4. Questions and Exercises

Quick Question 1:
   a. At what optical depth $\tau$ does the local temperature $T$ in a stellar atmosphere equal the stellar effective temperature $T_{\text{eff}}$?
   b. At about what optical depth $\tau$ does the local temperature $T = 10T_{\text{eff}}$?

Quick Question 2:
   a. Near the Sun’s surface where the temperature is at the effective temperature $T = T_* = T_{\text{eff}} \approx 5800$ K, compute the scale height $H$ (in km).
   b. Using the fact that the mean-free-path $\ell \approx H$ near this surface, compute the mass density $\rho$ (in g/cm$^3$) assuming the opacity is equal to the electron scattering value given in §D.1, i.e. $\kappa_e = 0.34$ cm$^2$/g.
15. Structure of Radiative vs. Convective Stellar Envelopes

15.1. $L \sim M^3$ relation for hydrostatic, radiative stellar envelopes

As discussed in part I, observations of binary systems show that main sequence stars follow an empirical mass-luminosity relation $L \sim M^3$. The physical basis for this can be understood by considering the two basic relations of stellar structure, namely hydrostatic equilibrium and radiative diffusion, as given in eqns. (13.1) and (14.7) above.

As in the Virial scaling for internal temperature given in §13.3, we can use a single point evaluation of the hydrostatic pressure gradient to derive a scaling between interior temperature $T$, stellar radius $R$ and mass $M$, and molecular weight $\mu$,

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}$$
$$\frac{T}{\rho \mu R} \sim \frac{M}{R^2}$$
$$TR \sim M \mu,$$  \hspace{1cm} (15.1)

Likewise, a single point evaluation of the temperature gradient in the radiative diffusion equation (14.7) gives

$$F = -\frac{16 \sigma sb T^3}{3 \kappa \rho} \frac{dT}{dr}$$
$$\frac{L}{R^2} \sim \frac{R^3 T^4}{\kappa M R}$$
$$L \sim \frac{(RT)^4}{\kappa M}$$
$$L \sim \frac{M^3 \mu^4}{\kappa},$$  \hspace{1cm} (15.2)

where the last scaling uses the hydrostatic equilibrium scaling in (15.1) to derive the basic scaling law $L \sim M^3$, assuming a fixed molecular weight $\mu$ and stellar opacity $\kappa$.

Two remarkable aspects of this derivation are that (1) the role of the stellar radius cancels; and (2) the resulting $M - L$ scaling does not depend on the details of the nuclear generation of the luminosity in the stellar core! Indeed, this scaling was understood from stellar structure analyses that were done (e.g. by Eddington, and Schwarzschild) in the
1920’s, long before Hydrogen fusion was firmly established as a key energy source for the sun and other main-sequence stars (e.g., by Hans Bethe ca. 1939).

### 15.2. Horizontal-track Kelvin-Helmholtz contraction to the main sequence

In fact, this simple $L \sim M^3$ scaling even applies to the final stages of pre-main-sequence evolution, when the core is not yet hot enough to start nuclear burning, but the envelope has become hot enough for radiative diffusion to dominate the transport of energy generated by the star’s gravitational contraction. As the radius decreases over the Kelvin-Helmholtz timescale $t_{KH}$ of this contraction, the surface temperature increases in a way that keeps the luminosity nearly constant. The lower panel of figure 15.1 illustrates that, on the H-R diagram, a late-phase pre-main-sequence star thus evolves along a horizontal track from right to left, stopping when it reaches the main sequence; this is where the core temperature is now high enough for H-fusion to take over in supplying the energy for the stellar luminosity, without any need for further contraction. As discussed in §16, for a given mass, a star’s radius on the main sequence is just the value for which the interior temperature, as set by the Virial theorem, is sufficiently high to allow this H-fusion in the core.

### 15.3. Convective instability and energy transport

In practice, the transport of energy from the stellar interior toward the surface sometimes occurs through convection instead of radiative diffusion, and this has important consequence for stellar structure and thus for the scaling of luminosity.

Convection refers to the overturning motions of the gas, much like the bubbling of boiling water on a stove. Stars become unstable to forming convection whenever the processes controlling the temperature make its spatial gradient too steep. This can occur in the nuclear burning core of massive stars, for which the specific mechanism for Hydrogen fusion, called the “CNO” cycle, gives the nuclear burning rate a steep dependence on temperature. The resulting steep temperature gradient makes the cores of such stars strongly convective.

Steep gradients, and their associated convection, can also occur in outer regions of cooler, lower-mass stars, where the cooler temperature induces recombination of ionized H or He. The bound-free absorption by this neutral Hydrogen significantly increases the local stellar opacity $\kappa$. For a fixed stellar flux $F = L/4\pi r^2$ of stellar luminosity $L$ that needs to be transported through an interior radius $r$, the radiative diffusion eqn. (14.7) shows that
Fig. 15.1.— Illustration of the pre-main-sequence evolution of stars. The upper panel shows how during the early stages of a collapsing proto-star, the interior is fully convective, causing it to evolve with decreasing luminosity at a nearly constant, relatively cool surface temperature, and so down the nearly vertical “Hayashi track” in the H-R diagram. The lower panel shows the final approach to the main sequence for stars of various masses. For stars with a solar mass or above, the stellar interior becomes radiative, stopping the Hayashi track decline in luminosity. The stars then evolve horizontally and to the left on the H-R diagram, each with fixed luminosity but increasing temperature, till they reach their respective positions on the “zero-age-main-sequence” or ZAMS, when the core is hot enough to ignite H-fusion.
the required radiative temperature gradient increases with such increased opacity,

\[
\left| \frac{dT}{dr} \right|_{\text{rad}} = \frac{3\kappa \rho F}{16\sigma_{sb}T^3} \sim \kappa
\]

(15.3)

Fig. 15.2.— Illustration of upward displacement of a spherical fluid element in test for convective instability, which occurs when the displaced element has a lower density \( \rho_1' \) than that of its surroundings, \( \rho_1 \). Since the pressure must remain equal inside and outside the element, this requires the element to have a higher temperature, \( T_1' > T_1 \). Since the overall temperature gradient is negative, convection thus occurs whenever the magnitude of the atmospheric temperature gradient is steeper than the adiabatic gradient that applies for the adiabatically displaced element, i.e., \( |dT/dr| > |dT/dr|_{\text{ad}} \). If this gradient becomes too steep, then, as illustrated in figure 15.2, a small element of gas that is displaced slightly upward becomes less dense than its surroundings, giving it a buoyancy that causes it to rise higher still. A key assumption is that this dynamical rise of the fluid occurs much more rapidly than the rate for energy to diffuse into or out of the gas element. Processes that occur without any such energy exchange with the surroundings are called “adiabatic”, with a fixed (power-law) relation of pressure with density or temperature. In a hydrostatic medium with a set pressure gradient, this implies a fixed adiabatic temperature gradient \( (dT/dr)_{\text{ad}} \).
Starting from an initial radius $r_0$ with equal density and temperature inside and outside some chosen fluid element (i.e., $\rho'_0 = \rho, T'_0 = T_0$), let us determine the density $\rho'_1$ of that element after it is adiabatically displaced to a slightly higher radius $r_1 = r_0 + \delta r$, where the ambient density is $\rho_1$. Since dynamical balance requires the element and its surrounding to still have equal pressure after the displacement (i.e., $P'_1 = P_1$), we have by the perfect gas law that $\rho'_1 T'_1 = \rho_1 T_1$. If this upward displacement $\delta r > 0$ makes the element buoyant, with lower density $\rho'_1$ than that of its surroundings $\rho_1$, then using this constant pressure condition, we can derive the condition for the temperature gradient required for the associated convective instability,

$$\frac{T_1}{T'_1} = \frac{\rho'_1}{\rho_1} < 1$$; Convective instability

$$T_0 + \Delta r (dT/dr)_{\text{rad}} = T_1 < T'_1 = T_0 + \Delta r (dT/dr)_{\text{ad}}$$

$$\left| \frac{dT}{dr} \right|_{\text{rad}} > \left| \frac{dT}{dr} \right|_{\text{ad}},$$ (15.4)

where since both temperature gradients are negative, the condition in terms of absolute value requires a reversal of the inequality.

We thus see that convection will ensue whenever the magnitude of the radiative temperature gradient exceeds that of the adiabatic temperature gradient.

Convection is an inherently complex, 3D dynamical process that generally requires elaborate computer simulations to model accurately. A heuristic, semi-analytic model called “mixing length theory” has been extensively developed, but it has serious limitations, especially near the stellar surface, where the lower density and temperature can make convective transport quite inefficient. By contrast, in the dense and hot stellar interior, once convection sets in, it is so efficient at transporting energy that it keeps the local temperature gradient very close to the adiabatic value above which it is triggered.

One can thus just presume that the temperature gradient in interior convection regions is at the adiabatic value.

15.4. Fully convective stars – the Hayashi track for proto-stellar contraction

In hot stars with $T > 10,000$ K, Hydrogen remains fully ionized even to the surface; since there then is no recombination zone to increase the opacity and trigger convection, the energy transport in their stellar envelopes is by radiative diffusion. In moderately cooler stars
like the sun (with $T_\odot \approx 6000$ K), Hydrogen recombination in a zone just somewhat below the surface induces convection, which thus provides the final transport of energy toward the surface; but since the deeper interior remains ionized and thus non-convective, the general scaling laws derived assuming radiative transport still roughly apply for such solar-type stars.

However, in much cooler stars, with surface temperatures $T \approx 3500 - 4000$ K, the Hydrogen recombination extends deeper into the interior; this and other factors keep the opacity high enough to make the entire star convectively unstable right down to the stellar core. Because convection is so much more efficient than radiative diffusion, it can readily bring to the surface any energy generated in the interior – whether produced by gravitational contraction of the envelope, or by nuclear fusion in the core. As such, fully convective stars can have luminosities that greatly exceed the value implied by the $L \sim M^3$ scaling law derived in §15.1 (see eqn. 15.2) for stars with radiative envelopes. As discussed in §17, this is a key factor in the high luminosity of cool giant stars that form in the post-main-sequence phases after the exhaustion of Hydrogen fuel in the core.

But it also helps explain the high luminosity of the very cool, early stage of pre-main-sequence evolution, when gravitational contraction of a large proto-stellar cloud is providing the energy to make the cloud shine as a proto-star. Once the internal pressure generated is sufficient to establish hydro-static equilibrium, its interior becomes fully convective, forcing the proto-star to have this characteristic surface temperature around $T \approx 3500 - 4000$ K.

At early stages the proto-star’s radius is very large, meaning it has a very large luminosity $L = \sigma_s b T^4 4\pi R^2$. As it contracts, it stays at this temperature, but the declining radius means a declining luminosity. As illustrated in figure 15.1, during this early phase of gravitational contraction, the proto-star thus evolves down a nearly vertical line in the H-R diagram, dubbed the “Hayashi” track, after the Japanese scientist who first discovered its significance.

Once the radius reaches a level at which the luminosity is near the value predicted by the $L \sim M^3$ law, the interior switches from convective to radiative, and so the final contraction to the main sequence makes a sharp turn to a horizontal track (sometimes called the “Henyey” track) with nearly constant luminosity but decreasing surface temperature. The luminosity of this track is set by the stellar mass, according to the $L \sim M^3$ law derived for stars with interior energy transport by radiative diffusion. The contraction is halted when the core reaches a temperature (derived in the next section) for H-fusion, which then stably supplies the luminosity for the main-sequence lifetime.

As detailed in §17, once the star runs out of Hydrogen fuel in its core, its post-main-sequence evolution effectively traces backwards along nearly the same track followed during
this pre-main-sequence, ultimately leading to the cool, red giant stars seen in the upper right of the H-R diagram.

The timescale analyses in part I (§10) show that nuclear fusion of Hydrogen into Helium provides a long-lasting energy source that we can associate with main sequence stars in the H-R diagram (§6.3). But what are the requirements for such fusion to occur in the stellar core? And how is this to be related to the luminosity vs. surface temperature scaling for main sequence stars in the HR diagram? In particular, how might this determine the relation between mass and radius? Finally, what does it imply about the lower mass limit for stars to undergo Hydrogen fusion?

Fig. 16.1.— The dominant reaction channel for the proton-proton chain that characterizes hydrogen fusion in the sun and other low-mass stars.
16.1. Core temperature for H-fusion

In stars of a few solar masses or less, Hydrogen fusion occurs through direct proton-proton collision, known thus as p-p ‘burning’. Figure 16.1 illustrates the most important of the detailed reaction channels, but the overall result is simply

\[ 4 \, ^1H^+ \rightarrow 4He^{+2} + \nu + 2e^+ + E_\gamma, \]  

where \( \nu \) represents a weakly interacting neutrino (which simply escapes the star). The \( 2e^+ \) represents two positively charged “anti-electrons”, or \textit{positrons}, which quickly annihilate with ordinary electrons, releasing \( \sim 2 \times 2 \times \frac{1}{2} \approx 2 \text{ MeV} \) of energy. The rest of the net \( \sim 4 \times 7 \text{MeV} \) in energy, representing the mass-energy difference between \( 4H \) vs. one \( He \), is released as high-energy photons (\( \gamma \)-rays) of energy \( E_\gamma \).

The essential requirement for such p-p fusion is that the thermal kinetic energy \( kT \) of the protons overcome the mutual repulsion of their positive charge \(+e\), to bring the protons to a close separation at which the strong nuclear (attractive) force is able take over, and bind the protons together. For a given temperature \( T \), the minimum separation \( b \) for two protons colliding head-on comes from setting this thermal kinetic energy equal to the electrostatic repulsion energy,

\[ kT = \frac{e^2}{b}. \]  

In particular, if we were to require that this minimum separation be equal to the size of a Helium nucleus, i.e. \( b \approx 1 \text{ fm} = 10^{-15} \text{ m} \), then from eqn. (16.2) we find that the required temperature is quite extreme, \( T \approx 1.7 \times 10^{10} \text{ K}! \)

Comparison with the virial scaling (13.9) shows this is more than a thousand times the characteristic virial temperature for the solar interior, \( T_{\text{int}} \approx 13 \text{ MK} \). As such, the closest distance \( b \) between protons in the interior core of the sun is actually more than a thousand times the size of the Helium nucleus, which is thus well outside the scale for operation of the strong nuclear force that keeps the nucleus bound.

The reason that nuclear fusion can nonetheless proceed at such a relatively modest temperature stems again from the uncertainty principle of modern quantum physics. Namely, a proton with thermal energy \( m_pv_{th}^2/2 = kT \) has an associated momentum \( p = m_pv_{th} = \sqrt{2m_pkT} \). Within quantum mechanics, it thus has an associated ‘fuzziness’ in position, characterized by its De Broglie wavelength \( \lambda \equiv h/p \), where \( h \) is Planck’s constant. If \( \lambda \gtrsim b \),

\[ ^2\text{In higher mass stars higher core temperatures, H-fusion is catalyzed by nuclear reactions of Hydrogen with Carbon, Nitrogen, and Oxygen through what is known as the CNO cycle.}
then there is a good probability that this waviness of protons will allow them to ‘tunnel’ through the electrostatic repulsion barrier between them, and so find themselves within a nuclear distance at which the strong attractive nuclear force can bind them. Setting $b = \lambda = h/(m_p v_{th})$ in eqn. (16.2), we can thus obtain an explicit expression for the proton thermal speed needed for nuclear fusion of Hydrogen$^3$,

$$v_{th,nuc} = \frac{2e^2}{h} = 690 \text{ km/s}.$$ \hspace{1cm} (16.3)

Two remarkable aspects of eqn. (16.3) are: (1) this thermal speed for H-fusion depends only on the fundamental physics constants $e$ and $h$, and (2) its numerical value is very nearly equal to the surface escape speed from the sun, $v_{esc} = \sqrt{2GM_\odot/R_\odot} = 618 \text{ km/s}$. Recalling the virial scaling (13.9) that says the thermal energy in the stellar interior is comparable to the gravitational binding energy, this means that given the solar mass $M_\odot$ the sun has adjusted to just the radius needed for the gravitational binding to give an interior temperature that is hot enough for Hydrogen fusion. For mean molecular weight $\bar{\mu} \approx 0.6m_p$, the mean thermal speed (16.3) implies a core temperature

$$T_{nuc} = \frac{\bar{\mu} v_{th,nuc}^2}{2k} = 1.2 \frac{m_p e^4}{k h^2} \approx 17 \text{ MK},$$ \hspace{1cm} (16.4)

which now is quite comparable to the interior temperature $T_{int,vir} \approx 13 \text{ MK}$ obtained by applying the virial scaling (13.9) to the sun.

### 16.2. Main sequence scalings for radius-mass and luminosity-temperature

If we were to naively apply these same scalings to stars with different masses, then it would suggest all stars along the main sequence should have the same, solar ratio of mass to radius, and thus that the radius should increase linearly with mass, $R \sim M$.

In practice, the radius-mass relation for main-sequence stars is somewhat sublinear,

$$R \sim M^{0.7}. \hspace{1cm} (16.5)$$

This can be understood by considering that the much higher luminosity of more massive stars, scaling as $L \sim M^3$, means that the core – within which the total fuel available scales

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$^3$I am indebted to Prof. D. Mullan for pointing out to me this remarkably simple scaling.
just linearly with stellar mass $M$—must have more vigorous nuclear burning\(^4\). The higher core temperature to drive such more vigorous H-fusion then requires by the Virial theorem that the mass to radius ratio of such stars must be somewhat higher than for lower mass stars like the sun.

Combining such a sublinear radius-mass scaling $R \sim M^{0.7}$ with the mass-luminosity scaling $L \sim M^3$ (eqn. 15.2) and the Stefan-Boltzmann relation $L \sim T^4 R^2$, we infer that luminosity should be a quite steep function of surface temperature along the main sequence, viz. $L \sim T^8$. While observed HR diagrams (like that plotted for nearby stars in part I) show the main sequence to have some complex curvature structure, a straight line with $\log L \sim 8 \log T$ does give a rough overall fit, thus providing general support for these simple scaling arguments.

### 16.3. Lower mass limit for hydrogen fusion: Brown Dwarf stars

These nuclear burning scalings can also be used to estimate a minimum stellar mass for Hydrogen fusion. Stars with mass below this minimum are known as Brown Dwarfs. A key new feature of these stars is that their cores become "electron degenerate", and so no longer follow the simple virial scalings derived above for stars in which the pressure is set by the ideal gas law. Electron degeneracy occurs when the electron number density $n_e$ becomes comparable to cube of the electron De Broglie wavenumber $k_e \equiv 2\pi/\lambda_e \equiv 1/\bar{\lambda}_e$,

\[
n_e \approx k_e^3 = \frac{1}{\lambda_e^3},
\]

with the electron thermal De Broglie (reduced) wavelength,

\[
\bar{\lambda}_e = \frac{\hbar}{p_e} = \frac{\hbar}{\sqrt{2m_e kT}},
\]

where $\hbar \equiv h/2\pi$, and the latter equality casts the electron thermal momentum $p_e$ in terms the temperature $T$ and electron mass $m_e$. Assuming a constant density $\rho = M/(4\pi R^3/3) \approx m_p n_e$, we can combine (16.6) and (16.7) with the nuclear temperature (16.4) and the Virial relation (13.9) to obtain a relation for the stellar mass at which a nuclear burning core should

\(^4\)Indeed, as already noted, in massive stars the standard, direct proton-proton fusion is augmented by a process called the CNO cycle, in which CNO elements act as a catalyst for H-fusion. Attaching protons to such more highly charged CNO nuclei requires a higher core temperature to overcome the stronger electrical repulsion, and this indeed obtains in such massive stars.
become electron degenerate,

\[ M_{\text{min, nuc}} = \sqrt{\frac{3/2}{4\pi^2}} \left(\frac{m_p}{m_e}\right)^{3/4} \frac{e^3}{G^{5/2} m_p^2} \approx 0.1 M_\odot. \]  

(16.8)

Stars with a mass below this minimum should not be able to ignite H-fusion, because electron degeneracy prevents their cores from contracting to a small enough size to reach the \(\sim 17 \text{MK} \) temperature (see eqn. 16.4) required for fusion. In practice, more elaborate computations indicate such Brown Dwarf stars have a limiting mass \( M_{BD} \lesssim 0.08 M_\odot \), just slightly below the simple estimate given in (16.8).

Note that, although this minimum mass for H-fusion is limited by electron degeneracy, the actual value is independent of Planck’s constant \( h \)! In effect, the role of \( h \) in the tunneling effect for H-fusion cancels its role in electron degeneracy.

### 16.4. Upper mass limit for stars: the Eddington Limit

Let’s next consider what sets the upper mass limit for observed stars. This is not linked to nuclear burning or degeneracy, but stems from the strong \( L \sim M^3 \) scaling of luminosity with mass, which, as noted in §15.1, follows from the hydrostatic support and radiative diffusion of the stellar envelope.

In addition to its important general role as a carrier of energy, radiation also has an associated momentum, set by its energy divided by the speed of light \( c \). The trapping of radiative energy within a star thus inevitably involves a trapping of its associated momentum, leading to an outward radiative force, or for a given mass, an outward radiative acceleration \( g_{\text{rad}} \), that can compete with the star’s gravitational acceleration \( g \). For a local radiative energy flux \( F \) (energy/time/area), the associated momentum flux (force/area, or pressure) is just \( F/c \). The material acceleration resulting from absorbing this radiation depends on the effective cross sectional area \( \sigma \) for absorption, divided by the associated material mass \( m \), as characterized by the opacity \( \kappa \),

\[ g_{\text{rad}} = \frac{\sigma F}{mc} = \frac{\kappa F}{c}. \]  

(16.9)

For a star of luminosity \( L \), the radiative flux at some radial distance \( r \) is just \( F = L/4\pi r^2 \). This gives the radiative acceleration the same inverse-square radial decline as the stellar gravity, \( g = GM/r^2 \), meaning that it acts as a kind of “anti-gravity”.

Sir Arthur Eddington first noted that, even for a minimal case in which the opacity just comes from free-electron scattering, \( \kappa = \kappa_e = 0.2(1 + X) \approx 0.34 \text{cm}^2 \text{g}^{-1} \) (with the numerical
value for standard (solar) Hydrogen mass fraction \( X \approx 0.7 \), there is a limiting luminosity, now known as the “Eddington luminosity”, for which the radiative acceleration \( g_{\text{rad}} = g \) would completely cancel the stellar gravity,

\[
L_{\text{Edd}} = \frac{4\pi G M c}{\kappa_e} = 3.8 \times 10^4 L_\odot \frac{M}{M_\odot}.
\] (16.10)

Any star with \( L > L_{\text{Edd}} \) is said to exceed the “Eddington limit”, since even the radiative acceleration from just scattering by free electrons would impart a force that exceeds the stellar gravity, thus implying that the star would no longer be gravitationally bound!

For main sequence stars that follow the \( L \sim M^3 \) scaling, setting \( L = L_{\text{Edd}} \) yields an estimate for an upper mass limit at which the star would reach this Eddington limit,

\[
M_{\text{max,Edd}} \approx 195 M_\odot.
\] (16.11)

where \( 195 \approx \sqrt{3.8 \times 10^4} \). This agrees quite well with modern empirical estimates for the most massive observed stars, which are in the range 150-300 \( M_\odot \).

Actually, as stars approach this Eddington limit, the radiation pressure alters the hydrostatic structure of the envelope, causing the mass-luminosity relation to weaken toward a linear scaling, \( L \sim M \), and so allowing in principle for stars with even with mass \( M > M_{\text{max,Edd}} \) to remain bound. In practice, such stars are subject to “photon bubble” instabilities, much as occurs whenever a heavy fluid (in this case the stellar gas) is supported by a lighter one (here the radiation). Very massive stars near this Eddington limit thus tend to be highly variable, often with episodes of large ejection of mass that effectively keeps the stellar mass near or below the \( M_{\text{max,Edd}} \approx 195 M_\odot \) limit.

**Excercise 1:**

a. Assume a power-law radius-mass scaling \( R \sim M^a \) for stars on the main-sequence. Show that there is an associated power-law relation \( L \sim T^b \) between luminosity \( L \) and surface temperature \( T \).

b. Give a formula for the power-index \( b \) in terms of the index \( a \).

c. Compute the values for \( b \) for cases with \( a = 0.5, 0.7 \) and 1.

d. What value of \( a \) would give the \( L \sim T^8 \) quoted in the text.
17. Post-Main-Sequence Evolution

As a star ages, more and more of the Hydrogen in its core becomes consumed by fusion into Helium. Once this core Hydrogen is exhausted, how does the star react and adjust? Without the H-fusion to supply its luminosity, one might think that perhaps the star would simply shrink, cool and dim, and so die out, much as a candle when all its wax is used up.

Instead it turns out that stars at this post-main-sequence stage of life actually start to expand, at first keeping roughly the same luminosity and so becoming cooler at the surface, but eventually becoming much brighter giant or supergiant stars, shining with a luminosity that can be thousands or even tens of thousands that of their core-H-burning main sequence.

Figure 17.1 illustrates the post-MS evolution for the sun (left) and stars with mass up to 10 $M_\odot$ (right). Let us focus our initial attention on solar type stars with $M \lesssim 8M_\odot$.

17.1. Post-MS evolution and death of solar-type stars with $M \lesssim 8M_\odot$

17.1.1. Core-Hydrogen burning and evolution to the Red Giant branch

The apparently counterintuitive post-main-sequence adjustment of stars can actually be understood through the same basic principals used to understand their initial, pre-main-
sequence evolution. When the core runs out of Hydrogen fuel, the lack of energy generation does indeed cause the core itself contract. But the result is to make this core even denser and hotter. Then, much as the hot coals at the heart of a wood fire help burn the wood fuel around it much faster, the higher temperature of a contracted stellar core actually makes the overlying shell of Hydrogen fuel around the core burn even more vigorously!

Now, unlike during the main sequence – when there is an essential regulation or compatibility between the luminosity generated in the core and the luminosity that the radiative envelope is able to transport to the stellar surface –, this shell-burning core is actually over-luminous relative to the envelope luminosity that is set by the \( L \sim M^3 \) scaling law. As such, instead of emitting this core luminosity as surface radiation, the excess energy acts to re-inflate the star, in effect doing work against gravity to reverse the Kelvin-Helmholtz contraction that occurred during the star’s pre-main-sequence evolution. Initially, the radiative envelope keeps the luminosity fixed so that, as the star expands, the surface temperature again declines, with the star thus again evolving horizontally on the H-R diagram, this time from left to right.

But as the surface temperature approaches the limiting value \( T \approx 3500 - 4000 \) K, the envelope again becomes more and more convective, which thus now allows this full high-luminosity of the H-shell-burning core to be transported to the surface. The star’s luminosity thus increases, with now the temperature staying nearly constant at the cool value for the Hayashi limit. In the H-R diagram, the star essentially climbs back up the Hayashi track, eventually reaching the region of the cool, red giants in the upper right of the H-R diagram.

The above describes a general process for all stars, but the specifics depend on the stellar mass. For masses less than the sun, the main sequence temperature is already quite close to the cool limit, so evolution can proceed almost directly vertically up the Hayashi track. For masses much greater than the sun, the luminosity and temperature on the main sequence are both much higher, and so the horizontal evolutionary phase is more sustained. And since the luminosity is already very high, these stars become red supergiants without ever having to reach or climb the Hayashi track.

17.1.2. Helium flash and core-Helium burning on the Horizontal Branch

This Hydrogen-shell burning also has the effect of increasing further the temperature of the stellar core, and eventually this reaches a level where the fusion of the Helium itself
becomes possible, through what’s known as the “triple-α process”\(^5\),
\[
3\, ^4\text{He}^+2 \rightleftharpoons ^8\text{Be}^+4 + ^4\text{He}^+2 \rightarrow ^{12}\text{C}^+6. \tag{17.1}
\]
The direct fusion of two \(^4\text{He}^+2\) nuclei initially make an unstable nucleus of Beryllium (\(^8\text{Be}^+4\)), which usually just decays back into Helium. But if the density and temperature are sufficiently high, then during the brief lifetime of the unstable Beryllium nucleus, another Helium can fuse with it to make a very stable Carbon nucleus \(^{12}\text{C}^+6\). Since the final step of fusing \(^4\text{He}^+2\) and \(^8\text{Be}^+4\) involves overcoming an electrostatic repulsion that is \(2 \times 4 = 8\) times higher than for proton-proton (p-p) fusion of Hydrogen, He-fusion requires a much higher core temperature, \(T_{He} \approx 120\) MK.

In stars with more than a few solar masses, this ignition of the Helium in the core occurs gradually, since the higher core temperature from the addition of He-burning increases the gas pressure, making the core tend to expand in a way that regulates the burning rate.

In contrast, for the sun and other stars with masses \(M < 2M_\odot\), the number density of electrons \(n_e\) in the helium core is so high\(^6\) that their core becomes electron degenerate. As discussed in §16.3 for the Brown dwarf stars that define the lower mass limit for H-burning, electron degeneracy occurs when the mean distance between electrons \(\sim n_e^{-1/3}\) becomes comparable to the DeBroglie wavelength \(\bar{\lambda}_e = \hbar/p_e\). The properties of such degeneracy are discussed further in §17.1.4 on the degenerate white-dwarf end states of solar-ype stars. But in the present context a key point is that, unlike for the ideal gas law, the pressure of a degenerate gas becomes \textit{independent of temperature}! This means that any heating of the core no longer causes a pressure-driven expansion that normally regulates the nuclear burning of a core governed by the ideal gas law.

Without then the regulation from such thermal pressure-driven expansion, the ignition of He burning leads to a \textit{Helium flash}, in which the entire degenerate core of Helium is fused into Carbon over a very short timescale. This flash marks the “tip” of the Red Giant Branch (RGB) in the H-R diagram, but somewhat surprisingly, the sudden addition of energy is largely absorbed by the overlying stellar envelope. The star thus quickly settles down to a more quiescent phase of He-burning. Because the He burning causes the core to expand, the

\(^5\)since Helium nuclei are sometimes referred as “α-particles”

\(^6\)Recall that on the main sequence the radii of stars is (very) roughly proportional to their mass, \(R \sim M\). But since density scales as \(\rho \sim M/R^3\), the density of low mass stars tends generally to be higher than in high-mass stars, roughly scaling as \(\rho \sim 1/M^2\). This overall scaling of average stellar density also applies to the relative densities of stellar cores, and so helps explain why the cores of low-mass stars tend to become electron degenerate, while those of higher mass stars do not.
shell burning of H actually declines, causing the luminosity to decrease from the tip of the Red Giant branch, where the He flash occurs, to a somewhat hotter, dimmer region known as the “Horizontal Branch” in the H-R diagram.

This Horizontal Branch (HB) can be loosely thought of as the He-burning analog of the H-burning Main Sequence (MS), but a key difference is that it lasts a much shorter time, typically only 10 to 100 million years, much less than the many billion years for a solar mass star on the MS. This is partly because the luminosity for HB stars is so much higher than for a similar mass on the MS, implying a much higher burn rate of fuel. But another factor is that the energy yield per-unit-mass, $\epsilon$, for He-fusion to Carbon is about a tenth of that for H-fusion to Helium, viz. about $\epsilon_{\text{He}} \approx 0.06\%$ vs. the $\epsilon_H \approx 0.7\%$ for H-burning. With lower energy produced, and higher rate of energy lost in luminosity, the lifetime is accordingly shorter.

17.1.3. Asymptotic Giant Branch to Planetary Nebula to White Dwarf

Once the core runs out of Helium, He-burning also shifts to an inner shell around the core, which itself is still surrounded by an outer shell of more vigorous H-burning. This again tends to increase the core luminosity, but now since the star is cool and thus mostly convective, this energy is mostly transported to the surface with only a modest further expansion of the stellar radius. This causes the star to again climb in luminosity along what’s called the “Asymptotic Giant Branch” (AGB), which parallels the Hayashi track at just a somewhat hotter surface temperature.

In the sun and stars of somewhat higher mass, up to $M \lesssim 8M_\odot$, there can be further ignition of the Carbon to fuse with Helium to form Oxygen. But further synthesis up the periodic table requires overcoming the greater electrical repulsion of more highly charge nuclei. This in turn requires a temperature higher than occurs in the cores lower mass stars, for which the onset of electron degeneracy prevents contraction to a denser, hotter core. Further core burning thus ceases, leaving the core as an inert, degenerate ball of C and O, with final mass on order of $1M_\odot$, with the remaining mass contained in the surrounding envelope of mostly Hydrogen.

But such AGB stars tend also to be pulsationally unstable, and because of the very low surface gravity, such pulsations can over time actually eject the entire stellar envelope. This forms a circumstellar nebula that is heated and ionized by the very hot remnant core. As seen in the left panel of figure 17.6, the resulting circular nebular emission glow somewhat resembles the visible disk of a planet, so these are called “planetary nebulae”, though they
really have nothing much to do with actual planets. After a few thousand years, the planetary nebula dissipates, leaving behind just the degenerate remnant core, a white dwarf star.

17.1.4. White Dwarf stars

The degenerate nature of white dwarf stars endows them with some rather peculiar, even extreme properties. As noted, they typically consist of roughly a solar mass of C and O, but have a radius comparable to that of the earth, \( R_e \approx R_\odot/100 \). This small radius makes them very dense, with \( \rho_{wd} \approx 10^6 \rho_\odot \approx 10^6 \text{ g/cm}^3 \), i.e. about a million times (!) the density of water, and so a million times the density of normal main-sequence stars like the sun. It also gives them very strong surface gravity, with \( g_{wd} \approx 10^4 g_\odot \approx 10^6 \text{ m/s}^2 \), or about 100,000 times earth’s gravity!

As noted in §16.3 for the Brown dwarf stars that define the lower mass limit for H-burning, gas becomes electron degenerate when the electron number density \( n_e \) becomes so high that the mean distance between electrons becomes comparable to their reduced De Broglie wavelength,

\[
    n_e^{-1/3} \approx \bar{\lambda} \equiv \frac{\hbar}{m_e v_e},
\]

where the electron thermal momentum \( p_e \) equals the product of its mass \( m_e \) and thermal speed \( v_e \), and \( \hbar \equiv h/2\pi \) is the reduced Planck constant. The associated electron pressure is

\[
    P_e = n_e v_e p_e = n_e^{5/3} \frac{\hbar^2}{m_e} = \left( \frac{\rho Z}{A m_p} \right)^{5/3} \frac{\hbar^2}{m_e},
\]

where the last equality uses the relation between electron density and mass density, \( \rho = n_e A m_p/Z \), with \( Z \) and \( A m_p \) the average nuclear charge and atomic mass. For example, for a Carbon white dwarf, the atomic number \( Z \) gives the number of free (ionized) electrons needed to balance the +Z charge of the Carbon nucleus, while the atomic weight \( A m_p \) gives the associated mass from the C atoms. The hydrostatic equilibrium (cf. eqn. 13.1) for pressure gradient support against gravity then requires for a white dwarf star with mass \( M_{wd} \) and radius \( R_{wd} \),

\[
    \frac{P_e}{R_{wd}} \approx \rho \frac{G M_{wd}}{R_{wd}^2}.
\]

Using the density scaling \( \rho \sim M_{wd}/R_{wd}^3 \), we can combine (17.3) and (17.4) to solve for a relation between the white-dwarf radius and its mass,

\[
    R_{wd} = \frac{1}{G M_{wd}^{1/3}} \frac{\hbar^2}{m_e} \left( \frac{Z}{A m_p} \right)^{5/3} \approx 0.01 R_\odot \left( \frac{M_\odot}{M_{wd}} \right)^{1/3},
\]

(17.5)
where the approximate evaluation uses the fact that for both C and O the ratio \( Z/A = 1/2 \). For a typical mass of order the solar mass, we see that a white dwarf is very compact, comparable to the radius of the earth, \( R_e \approx 0.01 R_\odot \). But note that this radius actually 
_decreases_ with increasing mass.

### 17.1.5. Chandrasekhar limit for white-dwarf mass: \( M < 1.4M_\odot \)

This fact that white-dwarf radii decrease with higher mass means that, to provide the higher pressure to support the stronger gravity, the electron speed \( v_e \) must strongly increase with mass. Indeed, at some point this speed approaches the speed of light, \( v_e \approx c \), implying that the associated electron pressure now takes the scaling (cf. eqn. 17.3),

\[
P_e = n_e c p_e = n_e^{4/3} \hbar c = \left( \frac{\rho Z}{A m_p} \right)^{4/3} \hbar c.
\]

Applying this in the hydrostatic relation (17.4), we now find that the radius \( R \) cancels, and we instead can solve for a _upper limit_ for a white dwarf’s mass,

\[
M_{wd} \leq M_{ch} = \sqrt{\frac{3}{2\pi}} \left( \frac{\hbar c}{G} \right)^{3/2} \left( \frac{Z}{A m_p} \right)^2 \approx 1.4M_\odot,
\]

where the subscript refers to “Chandrasekhar”, the astrophysicist who first derived this mass limit, and the proportionality factor \( \sqrt{\frac{3}{2\pi}} \) comes from a detailed calculation beyond the scope of the discussion here.

As discussed in later sections, when accretion of matter from a binary companion puts a white dwarf over this limit, it triggers an enormous “white-dwarf supernova” explosion, with a large, well-defined peak luminosity, \( L \approx 10^{10} L_\odot \). This provides a very bright standard candle that can be used to determine distances as far as a Gpc, giving a key way to calibrate the expansion rate of the universe.

But in the present context, this limit means that sufficiently massive stars with cores above this mass cannot end their lives as a white dwarf. Instead, they end as violent “core-collapse supernovae”, leaving behind an even more compact final remnant, either a neutron star or black hole, as we discuss next.
17.2. Post-MS evolution and death of high-mass stars with $M > 8M_\odot$

17.2.1. Multiple shell burning and horizontal loops in H-R diagram

The post-main-sequence evolution of stars with higher initial mass, $M > 8M_\odot$ has some distinct differences from that outlined above for solar and intermediate mass stars. Upon exhaustion of H-fuel at the end of the main sequence, such stars again expand in radius because of the over-luminosity of H-shell burning. But the high luminosity and high surface temperature on the main sequence means that their stellar envelopes remain radiative even as they expand, never reaching the cool temperatures that force a climb up the Hayashi track. Instead, their evolution tends to keep near the constant luminosity set by $L \sim M^3$ scaling for the star’s mass, so evolving horizontally to the right on the H-R diagram.

Since stellar radii scale nearly linearly with mass $R \sim M$, the mean stellar density $\rho \sim M/R^3 \sim M^{-2}$ tends to decline with increasing mass. Thus, even after the core contraction that occurs toward the end of nuclear burning, the core density of high-mass stars never becomes high enough to become electron degenerate. Moreover the higher mass means a stronger gravitational confinement that gives higher central temperature and pressure. This now makes it possible to overcome the increasingly strong electrical repulsion of more highly charged, higher elements, allowing nucleosynthesis to proceed up the period table all the way to Iron, which is the most stable nucleus.

However, each such higher levels of nucleosynthesis yields proportionally less and less energy. This can be seen from the plot in figure 17.2 of the binding energy per nucleon vs. the number of nucleons in a nucleus. The jump from H to He yields 7.1 MeV, which relative to a nucleon mass of 931 Mev represents a percentage energy release of about 0.7%, as noted above. But from He to C the release is just 7.7 – 7.1 = 0.6 MeV, representing an energy efficiency of just 0.06%. As the curve flattens out, the fractional energy release become even less, until for elements beyond Iron, further fusion would require the addition of energy.

For such massive stars, the final stages of post-main sequence evolution are characterized by an increasingly massive Iron core that can no longer produce any energy by further fusion. But fusion still occurs in a surrounding series of shells, somewhat like an onion skin, with higher elements fusing in the innermost, hottest shells, and outer shells fusing lower elements, extending to an outermost shell of H-burning (see figure 17.3).
The fusion of light elements moves nuclei to the right, releasing the energy of nuclear burning in the very hot dense cores of stars, but only up to formation of the most stable nucleus, for iron (Fe), with atomic number $A = 56$. In the sudden core collapse of massive-star supernovae, the copious energy available synthesizes elements even beyond iron. The fission of such heavy elements leads to lower-mass nuclei toward the left. The energy released is what powers nuclear fission reactors here on earth.

17.2.2. Core-collapse supernovae

With the build-up of Iron in the core, there is an increasingly strong gravity, but without the further fusion-generated energy to keep the temperature high, the core pressure becomes unable to support the mass above. This eventually leads to a catastrophic core collapse, halted only when the electrons merge with the protons in the Iron nuclei to make the entire core into a collection of neutrons, with a density so high that they now actually become neutron degenerate. The “stiffness” of this neutron-degenerate core leads to a “rebound” in the collapse, with gravitational release from the core contraction now powering an explosion that blows away the entire outer regions of the star, with the stellar ejecta reaching speeds of about 10% the speed of light! This ejecta contains Iron and other heavy elements, including
Fig. 17.3.— The “onion-skin” layering of the core of a 25-$M_\odot$ star just before supernovae core collapse, illustrating the various stages of nuclear burning in shells around the inert iron core. The right box shows the decreasing duration for each higher stage of burning.

even those beyond Iron that are fused in less than a second of the explosion by the enormous energy and temperatures. While elements up to Oxygen can also be synthesized in low-mass stars, all the heavier elements in our universe, including here on earth, originated in supernova explosions. For a few weeks, the luminosity of such a supernova can equal or exceed that of a whole galaxy, up to $\sim 10^{12}L_\odot$!

Though the dividing line is not exact, it is thought that all stars with initial masses $M > 8M_\odot$ will end their lives with such a core-collapse supernova, instead of following the track, AGB $\rightarrow$ PN $\rightarrow$ White Dwarf, for stars with initial mass $M < 8M_\odot$. Stars with initial masses $8M_\odot < M \lesssim 30M_\odot$ are thought to leave behind a neutron star remnant, as discussed next. But we shall also see that such neutron star remnants have their own upper mass limit of $M_{ns} \lesssim 3M_\odot$, beyond which the gravity becomes so strong that not even the degenerate pressure from neutrons can prevent a further collapse, this time forming a black hole. This is thought to be the final core remnant for the most massive stars, those with initial mass $M \gtrsim 30M_\odot$. 

---

<table>
<thead>
<tr>
<th>Stage</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>H $\rightarrow$ He</td>
<td>7x10^6 years</td>
</tr>
<tr>
<td>He $\rightarrow$ C</td>
<td>7x10^5 years</td>
</tr>
<tr>
<td>C $\rightarrow$ O</td>
<td>600 years</td>
</tr>
<tr>
<td>O $\rightarrow$ Si</td>
<td>6 months</td>
</tr>
<tr>
<td>Si $\rightarrow$ Fe</td>
<td>1 day</td>
</tr>
<tr>
<td>Core Collapse</td>
<td>1/4 second</td>
</tr>
</tbody>
</table>
17.2.3. Neutron stars

Neutron stars are even more bizarrely extreme than white dwarfs. With a mass typically about twice the sun’s, they have a radius comparable to a small city, \( R_{ns} \approx 10 \, \text{km} \), about a factor 600 smaller than even a white dwarf, implying a density that is about \( 10^8 \) higher, and a surface gravity more than \( 10^5 \) times higher.

The radius and mass properties associated with neutron degeneracy can be derived by a procedure completely analogous to that used in §§17.1.4 and 17.1.5 for white dwarfs, just substituting now the electron mass with the neutron mass, \( m_e \rightarrow m_n \approx m_p \), and setting \( Z/A = 1 \). The radius-mass relation thus now becomes (cf. eqn. 17.5),

\[
R_{ns} = 2^{5/3} \frac{m_e}{m_p} R_{wd} = \frac{1}{GM_{ns}} \frac{\hbar^2}{m_n^{8/3}} \approx 10 \, \text{km} \left( \frac{M_{\odot}}{M_{ns}} \right)^{1/3}.
\]

(17.8)

Note again that, as in the case of an electron-degenerate white dwarf, this neutron-star radius also decreases with increasing mass.

For the same reasons that lead to the upper mass limit for white dwarfs, for sufficiently high mass the neutrons become relativistic, leading now to an upper mass limit for neutron stars (cf. eqn. 17.7) that scales as

\[
M_{ns} \leq M_{\text{lim}} = 1.6 \left( \frac{\hbar c}{G} \right)^{3/2} \left( \frac{1}{m_p} \right)^2 \approx 3M_{\odot},
\]

(17.9)

where again the factor 1.6 comes from detailed calculations not covered here; apart from this and the factor \( (\sim A/Z)^2 = 4 \), this is the same form as the Chandrasekhar mass for white dwarfs in eqn. 17.7. Neutron stars above this mass will again collapse, this time forming a black hole.

17.2.4. Black Holes

Black holes are objects for which the gravity is so strong that not even light itself can escape. A proper treatment requires General Relativity, Einstein’s radical theory of gravity that supplants Newton’s theory of universal gravitation, and extends it to the limit of very strong gravity. But we can nonetheless use Newton’s theory to derive some basic scalings. In particular, for a given mass \( M \), a characteristic radius for which the Newtonian escape speed is equal to speed of light, \( v_{esc} = c \), is just

\[
R_{bh} = \frac{2GM}{c^2} \approx 3 \, \text{km} \frac{M}{M_{\odot}},
\]

(17.10)
which is commonly known as the “Schwarzschild radius”.

Since the speed of light is the highest speed possible, any object within this Schwarzschild radius of a given mass $M$ can never escape the gravitational binding with that mass. In terms of Einstein’s General Theory of Relativity, mass acts to bend space and time, much the way a bowling ball bends the surface of a trampoline. And much as a sufficiently dense, heavy ball could rip a hole in the trampoline, for objects with mass concentrated within a radius $R_{\text{bh}}$, the bending becomes so extreme that it effectively punctures a hole in space-time. Since not even light can ever escape from this hole, it is completely black, absorbing any light or matter that falls in, but never emitting any light from the hole itself. This the origin of the term “black hole”.

Stellar-mass black holes with $M \gtrsim 3M_\odot$ form from the deaths of massive stars. If left over from a single star, they are hard or even impossible to detect, since by definition they don’t emit light.

However, in a binary system, the presence of a black hole can be indirectly inferred by observing the orbital motion (visually or spectroscopically via the Doppler effect) of the luminous companion star.

Moreover, when that companion star becomes a giant, it can, if it is close enough, transfer mass onto the black hole. Rather than falling directly into the hole, the conservation of the angular momentum from the stellar orbit requires that the matter first feed an orbiting accretion disk. By the Virial theorem, half the gravitational energy goes into kinetic energy of orbit, but the other half is dissipated to heat the disk, which by the blackbody law then emits it as radiation.

The luminosity of such black-hole accretion disks can be very large. For a black hole of mass $M_{\text{bh}}$ accreting mass at a rate $\dot{M}_a$ to a radius $R_a$ that is near the Schwarzschild radius $R_{\text{bh}}$, the luminosity generated is

$$L_{\text{disk}} = \frac{GM_{\text{bh}}\dot{M}_a}{2R_a} = \frac{R_{\text{bh}}}{4R_a} \dot{M}_ac^2 \equiv \epsilon \dot{M}_ac^2.$$  \hspace{1cm} (17.11)

The latter two equalities define the efficiency $\epsilon \equiv R_{\text{bh}}/4R_a$ for converting the rest-mass-energy of the accreted matter into luminosity. For accretion radii approaching the Schwarzschild radius, $R_a \approx R_{\text{bh}}$, this efficiency can be as high as $\epsilon \approx 0.25$, implying 25% of the accreted matter-energy is converted into radiation. By comparison, for H-fusion of a MS star the overall conversion efficiency is about 0.07%, representing the $\sim 10\%$ core mass that is sufficiently hot for H-fusion at a specific efficiency, $\epsilon_H = 0.007 = 0.7\%$. 
Quick Question 1:
(a) Because of general relativistic effects, it turns out the lowest stable orbit around a black hole is at radius of $3R_{\text{bh}}$. What is the luminosity efficiency for accreting to this radius?
(b) What is the accretion luminosity, in $L_{\odot}$, for a mass accretion rate $\dot{M}_a = 10^{-6}M_{\odot}/\text{yr}$ to this radius?
(c) Challenge problem: For a black hole with mass $M_{\text{bh}} = 3M_{\odot}$, use the Stefan-Boltzmann law to derive the radiative flux that would balance the local gravitational heating at this radius $r = 3R_{\text{bh}}$, and then solve for the local black-body temperature $T(r = 3R_{\text{bh}})$. Express this first as a ratio to sun’s surface temperature $T_{\odot}$, and then also in Kelvin.

Exercise 1: Use Wien’s law to compute the peak wavelength (in nm) of thermal emission from the inner region of an accretion disk with temperature $T = 10^7\text{ K}$. What is the energy (in eV) of a photon with this wavelength? Now also answer both questions for $T = 10^{10}\text{ K}$. What parts of the electromagnetic spectrum do these photon wavelengths/energies correspond to?

In the inner disk, the associated blackbody temperature can reach $10^7\text{ K}$ or more (see challenge problem); and at the very inner disk edge, dissipation of the orbital energy can
heat material to even more extreme temperatures, up to $\sim 10^{10}$ K. The associated radiation is very high energy, as seen from the Quick-Question calculations. By studying this high-energy radiation, we can infer the presence and basic properties (mass, even rotation rate) of black holes in such binary systems, even though we can’t see the black hole itself.

Figure 17.4 shows an artist depiction of the mass transfer accretion in the high-mass X-ray binary Cygnus X-1, thought to be the clearest example of a stellar-remnant black hole, estimated in this case to have a mass $M_{bh} > 10M_{\odot}$ that is well above the $M_{\text{lim}} \approx 3M_{\odot}$ upper limit for a neutron star (cf. eqn. 17.9).

Fig. 17.5.— Left: Optical image of the Crab Nebula, showing the remnant from a core-collapse supernova whose explosion was observed by Chinese astronomers in 1054. Right: A composite zoomed-in image of the central region Crab Nebula, showing the optical (red) image superimposed with an X-ray (blue) image made by NASA’s Chandra X-ray observatory. The bright star at the nebular center is the Crab pulsar, a rapidly rotating neutron star that was left over from the supernova explosion.

### 17.3. Observations of stellar remnants

It is possible to observe directly all three types of stellar remnants:

1. *Planetary Nebula and White Dwarf stars*
Fig. 17.6.— Left: M57, known at the Ring Nebula, provides a vivid example of a spherically symmetric Planetary Nebula. The central hot star is the remnant of the stellar core, and after the nebula dissipates, it will be left as a White Dwarf star. Right: A gallery of planetary nebulae, showing the remarkable variety of shapes that probably stem from interaction of the stellar ejecta with a binary companion, or perhaps even with the original star’s planetary system.

Stars with initial mass $M < 8M_\odot$ evolve to an AGB star that ejects the outer stellar envelope to form a Planetary Nebula (PN) with the hot stellar core with mass below the Chandrasek mass, $M_{wd} < M_{ch} = 1.4M_\odot$. Once the nebula dissipates, this leave behind White Dwarf (WD). White dwarf stars are very hot, but with such a small radius that their luminosity is very low, placing them on the lower left of the H-R diagram.

The excitation and ionization of the gas in the surrounding PN makes it shine with an emission line spectrum, with the wavelength-specific emission of various ion species giving it range of vivid colors or hues. Figure 17.6 shows that these PN can thus be visually quite striking, with spherical emission nebula from single stars (left), or very complex geometric forms (right) for stars in binary systems.

2. Neutron stars and Pulsars

Stars with initial masses in the range $8M_\odot < M \lesssim 30M_\odot$ end their lives as a core collapse supernova that leaves behind a neutron star with mass $1.4M_\odot < M_{ns} < 3M_\odot$. The conservation of angular momentum during the collapse to such a small size ($\sim 10$ km) makes them rotate very rapidly, often many times a second! This also generates
a strong magnetic field, and when the polar axis of this field points toward earth, it leads to a strong \textit{pulse} of beamed radiation in the radio to optical to even X-rays. This is observed as a \textit{pulsar}.

One of the best known examples is the Crab pulsar, which lies at the center of the Crab Nebula, the remnant from a core-collapse supernova that was observed by Chinese astronomers in 1054 AD. Figure 17.5 shows images of this Crab nebula in the optical region (left) and in a composite of images (right) in the optical (red) and X-ray (blue) wavebands.

3. \textit{Black holes and X-ray binary systems}

Finally, stars with initial masses $M \gtrsim 30M_\odot$ end their lives with a core collapse supernova that now leaves behind a black hole with mass $M_{bh} > 3M_\odot$. As noted, in single stars, these are difficult or impossible to observe, because they emit no light; but in binary systems, accretion from the other star can power a bright accretion disk around the black hole that radiates in high-energy bands like X-rays and even $\gamma$-rays. Figure 17.4 shows an artist depiction of accretion onto a black hole in the high-mass X-ray binary Cygnus X1.
C. Radiative Transfer

![Radiative Transfer Diagram]

Fig. C.1.— Emergent intensity from a semi-infinite, planar atmosphere. Along a direction \( \hat{s} \) that has projection \( \mu = \hat{s} \cdot \hat{r} = \cos \theta \) to the local vertical (radial) direction \( \hat{r} \), the change in intensity in each differential layer \( dr \) depends on thermal emission of radiation by the local Planck minus the absorption of local intensity \( I \), multiplied by the projected change in optical depth \(-d\tau/\mu = \kappa \rho ds\) along the path segment \( ds \).

C.1. Absorption and thermal emission in a stellar atmosphere

As noted above (§13.2), the atmospheric transition between interior and empty space occurs over a quite narrow layer, a few scale heights \( H \) in extent, which typically amounts to about a thousandth of the stellar radius (cf. eqn. 13.5). At any given location on the spherical stellar surface, the transport of radiation through this atmosphere can be modeled by treating it as a nearly planar layer, as illustrated in figure C.1.

To quantify this atmospheric transition between random-walk diffusion of the deep interior to free-streaming away from the stellar surface, we must now solve a differential equation that accounts for the competition between the reduction in intensity due to absorption vs. the production of intensity due to the local thermal emission \( B(\tau) \). As illustrated in figure C.1, consider a planar atmosphere with an arbitrarily large optical depth (at bottom) seen from an observer at optical depth zero (at top) who looks along a direction \( \hat{s} \) that has a pro-
jection \( \mu = \cos \theta \) to the local vertical (radial) direction \( \hat{r} \). The change in intensity in each differential layer \( dr \) depends on thermal emission of radiation by the local Planck function \( B \) minus the absorption of local intensity \( I \), multiplied by the projected change in optical depth \( -d\tau/\mu = \kappa \rho ds \) along the path segment \( ds \). This leads to an “equation of radiative transfer”,

\[
\frac{\mu dI(\mu, \tau)}{d\tau} = I(\mu, \tau) - B(\tau),
\]

where the radial optical depth integral is now defined from a distant observer at \( r \to \infty \),

\[
\tau(r) \equiv \int_r^\infty \kappa \rho dr',
\]

which thus places the observer at \( \tau(r \to \infty) = 0 \).

Fig. C.2.— Visible light picture of the solar disk, showing the center to limb darkening of the surface brightness.

\footnote{This standard notation using \( \mu \) for direction cosine here should not be confused with the notation in the previous sections that use \( \mu \) for molecular weight.}
C.2. The Eddington-Barbier relation for emergent intensity

Eqn. (C1) is a linear, first-order differential equation. As discussed in the exercise below, by using integrating factors, it can be converted to a formal integral solution for the emergent intensity seen by an external observer viewing the atmosphere along a projection $\mu$ with the local radius,

$$I(\mu, \tau = 0) = \int_0^\infty B(\tau) e^{-\tau/\mu} \frac{d\tau}{\mu} \approx B(\tau = \mu).$$  \hspace{1cm} (C3)

The latter approximation here assumes the Planck function is roughly a linear function of optical depth near the star’s surface, $B(\tau) \approx a + b\tau$. This so-called “Eddington-Barbier relation” states that when you peer into an opaque radiating gas, the emergent intensity you perceive is set by the value of the blackbody function at the location of unit optical depth along that ray. This in turn is set by the temperature at that location, providing a more rigorous definition for what we’ve referred to up to now as surface brightness and surface temperature.

An example of this E-B relation comes from the observed “limb darkening” of the solar disk, as illustrated by the visible light picture of the sun in fig. C.2. Because the line of sight looking at the center is more directly radial to the sun’s local surface, one can see into a deeper, hotter layer than from the more oblique angle when viewing toward the edge or “limb” of the solar disk. This makes the disk appear brightest at the center, and darker as the view moves toward the solar limb. The observed variation from center to limb thus provides a diagnostic of the temperature gradient in the sun’s surface layers.

Since stars are too far away to resolve their angular size, we can’t observe their emergent intensity $I(\mu, 0)$, but we can observe the flux $F(r) = L/4\pi r^2$ associated with the total luminosity $L = 4\pi R^2 \sigma_{sb} T_{\text{eff}}^4$. The emergent surface flux $F_* = L/4\pi R^2$ is obtained by integrating

---

8Of course, the brightness of the sun means we need special filters to see this effect. One should never look at the sun with the naked eye.
\[ \mu I(\mu, 0) \text{ over the } 2\pi \text{ solid angle for the hemisphere open to empty space, giving} \]

\[
F_* \equiv 2\pi \int_0^1 \mu I(0, \mu) d\mu \\
\approx 2\pi \int_0^1 \mu B(\tau = \mu) d\mu \\
\approx 2\pi \int_0^{1/3} \mu (a + b\mu) d\mu \\
= \frac{\pi B(\tau = 2/3)}{4/9} \\
= \frac{\sigma_{sb} T^4(\tau = 2/3)}{4/9}, \tag{C4}
\]

where the third equality assumes the Planck function near the surface can be approximated at a linear function of optical depth, \( B(\tau) \approx a + b\tau \).

Comparison of the final form of (C4) with the simple discussion of surface flux in part I shows that we can identify what we’ve been calling the stellar “surface” as the layer where the optical depth \( \tau(R) \equiv 2/3 \), with the “surface temperature” likewise just the temperature at this layer.

Stars are not really black-bodies, but it is convenient to define a star’s “effective temperature” \( T_{\text{eff}} \) as the blackbody temperature that would give the star’s inferred surface flux \( F_* = L/4\pi R^2 \). From (C4), we see that we can associate this effective temperature with the surface temperature at optical depth \( 2/3 \), \( T_{\text{eff}} = T(\tau = 2/3) \).

\section*{C.3. Questions and Exercises}

**Exercise 1:** Derive the integral solution (C3) from the differential equation (C1), assuming a semi-infinite atmosphere that extends to large depths \( \tau \to \infty \). Hint: First multiply eqn. (C1) by an integrating factor \( e^{-\tau/\mu} \), and use this to write the change in intensity in terms of a full differential. Then carry out the integral from the observer at \( \tau = 0 \) to some finite depth \( \tau \) where the intensity is taken to have a given value \( I(\tau, \mu) \). Finally take the limit \( \tau \to \infty \) to obtain (C3).

**Exercise 2:** If thermal emission from the Planck function is a linear function of radial optical depth \( B(\tau) = a + b\tau \), explicitly do the integration in (C3) to derive the Eddington-Barbier relation for emergent intensity \( I(\mu, 0) = B(\tau = \mu) \).
D. Atomic origins of opacity

For solid objects in our everyday world, the interaction with light depends on the object’s physical projected area, which is the source of the above concept of a “cross section”. But as noted in §12.3, for interstellar dust with sizes become comparable to the wavelength of light, the effective cross section can depend on this wavelength, and so differ from the projected geometric area.

For atoms, ions and electrons that make up a gaseous object like a star, the effective cross sections for interaction with light can be even more sensitive to the details. But generally because light is an Electro-Magnetic (EM) wave, at the atomic level its fundamental interaction with matter occurs through the variable acceleration of charged particles by the varying electric field in the wave. As the lightest common charged particle, electrons are most easily accelerated, and thus are generally key in setting the interaction cross section. The simplest example is that of an isolated free electron, so let’s begin by examining its interaction cross section and opacity.

D.1. Thomson cross-section and opacity for free electron scattering

As illustrated in the top left panel of figure D.1, when a passing EM wave causes a free electron to oscillate, it generates a wiggle in the electron’s own electric field, which then propagates away – at the speed of light – as a new EM wave in a new direction. Because an isolated electron has no way to store both the energy and momentum of the incoming light, it cannot by itself absorb the photon, and so instead simply scatters, or redirects it. The overall process is called “Thomson scattering”.

For such free electrons, the associated Thomson cross section can actually be accurately computed using the classical theory of electromagnetism. Intuitively, the scaling can be roughly understood in terms of the so-called “classical electron radius” \( r_e = e^2/m_ec^2 \), which is just the radius at which the electron’s electrostatic self-energy \( e^2/r_e \) equals the electron’s rest-mass energy \( m_ec^2 \). In these terms, the Thomson cross section for free-electron scattering is just a factor\(^9\) 8/3 times greater than the projected area of a sphere with the classical electron radius,

\[
\sigma_{Th} = \frac{8}{3} \pi r_e^2 = \frac{8}{3} \frac{\pi e^4}{m_e^2 c^4} = 0.66 \times 10^{-24} \text{cm}^2.
\]

\(^9\)This factor 8/3 comes from detailed classical calculations, and is not easy to understand in simple intuitive terms.
Fig. D.1.— Illustration of the free-electron and bound-electron processes that lead to scattering, absorption, and emission of photons.

For stellar material to have an overall neutrality in electric charge, even free electrons must still be associated with corresponding positively charged ions, which have much greater mass. Defining then a mean mass per free electron $\mu_e$, we can also define an electron scattering opacity $\kappa_e \equiv \sigma_{Th}/\mu_e$. Ionized hydrogen gives one proton mass $m_p$ per electron, but for fully ionized Helium (and indeed for most all heavier ions), there are two nucleon masses (one proton and one neutron, $m_p + m_n \approx 2m_p$) for each electron. For ionized stellar material with hydrogen mass fraction $X$, we thus have $\mu_e = 2m_p/(1 + X)$, which then gives for the opacity,

$$\kappa_e \equiv \frac{\sigma_{Th}}{\mu_e} = 0.2 (1 + X) \text{cm}^2/\text{g} = 0.34 \text{cm}^2/\text{g},$$

where the last equality assumes a “standard” solar Hydrogen mass fraction $X = 0.72$. 
D.2. Atomic absorption and emission: free-free, bound-bound, bound-free

When electrons are bound to atoms or ions, or even just nearby ions, then the combination of the electron and atom/ion can lead to true absorption of a photon of light. As shown in the center top row of figure D.1, for free electrons near ions, the shift in the electron trajectory as it passes an ion can now absorb a photon’s energy, a process called free-free absorption. The right top panel shows that the inverse process can actually produce a photon, and so is called free-free emission.

The second row illustrates bound-bound processes, involving up/down jumps of electrons between two bound energy levels of atom, with associated absorption/emission of photon (middle and right panel in second row), or indeed, a scattering if the absorption is quickly followed by a reemission of a photon with the same energy, but in a different direction (left panel, second row).

These bound-bound processes only work with photons with just the right energy to match the difference in energy levels, and so lead to the spectral line absorption or emission discussed earlier. But for those “just right” photons, the interaction cross section (leading to the opacity) can be much much higher than for Thomson scattering or free-free absorption, because in effect it is a kind of “resonance” interaction. An everyday analogy is blowing into a whistle vs. just into open air. In open air, you get a weak white noise sound, made up of a range of sound frequencies/wavelengths. With a whistle, the sound is loud and has a distinct pitch, representing a resonance oscillation at some well-defined frequency/wavelength.

The third row illustrates the bound-free processes associated with a photon absorption that causes an atom or ion to become (further) ionized by kicking off its electron. As with electron scattering or even free-free absorption, it is a continuum (vs. line) process, though it does now require that the photons have a energy equal to or greater than the ionization energy for that atom or ion. Its interaction cross-section can be significantly higher than electron scattering or free-free absorption, but is generally not as strong as for bound-bound processes that lead to lines.

The cross sections, and corresponding opacities, associated with these electron+ion/atom processes are much more complicated than for free electrons, and so are difficult to cast in the kind of simple formula given in eqn. D2 for Thomson electron scattering opacity. But often bound-free and free-free opacities are taken to follow a so-called “Kramer’s opacity” for which

\[ \kappa_{kr} \sim \rho T^{-7/2} \sim (P_{gas}/P_{rad}) T^{-1/2}. \]  

As discussed further below, often in stellar interiors the ratio of gas to radiation pressure is nearly constant, so that opacity decreases only weakly (as \(1/\sqrt{T}\)) with the increasing
temperature of the interior.

Some explicit opacity scaling formulae are given by:

www.astro.psu.edu/users/rbc/a534/lec6.pdf

A simple rough rule of thumb is that, outside of ionization zones where bound-free absorption can substantially enhance the overall opacity, stellar interiors typically have opacities that are some modest factor few times the simple electron scattering opacity in (D2), i.e., with a characteristic CGS value of order unity, $\kappa \approx 1 \text{ cm}^2/\text{g}$.

D.3. Questions and Exercises

Quick Question 1:

a. Seen standing up, what is the cross section (in cm$^2$) of a person with height 1.8 m and width 0.5 m?
b. If this person has a mass of 60 kg, what is his/her “opacity” $\kappa = \sigma/m$, in cm$^2$/g?
c. How does this compare with the typical opacity of stellar material?