7.1. Newton’s law of gravitation and stellar surface gravity

On Earth, an object of mass $m$ has a weight given by

$$W = F_{\text{grav}} = mg_e,$$  \hspace{1cm} (7.1)

where the acceleration of gravity on Earth is $g_e = 980 \text{ cm/s}^2 = 9.8 \text{ m/s}^2$. But this comes from Newton’s law of gravity, which states that for two point masses $m$ and $M$ separated by a distance $r$, the attractive gravitational force between them is given by

$$F_{\text{grav}} = \frac{GMm}{r^2},$$  \hspace{1cm} (7.2)

where Newton’s constant of gravity is $G = 6.7 \times 10^{-8} \text{ cm}^3/\text{g/s}$. Remarkably, when applied to spherical bodies of mass $M$ and finite radius $R$, the same formula works for all distances $r \geq R$ at or outside the surface! Thus, we see that the acceleration of gravity at the surface of the Earth is just given by the mass and radius of the Earth through

$$g_e = \frac{GM_e}{R_e^2}. \hspace{1cm} (7.3)$$
7.3. Speed for circular orbit

Let us next compare this escape speed with the speed needed for an object to maintain a circular orbit at some radius $r$ from the center a gravitating body of mass $M$. For an orbiting body of mass $m$, we require that the gravitational force be balanced by the centrifugal force from moving along the circle of radius $r$,

$$\frac{GMm}{r^2} = \frac{mV_{orb}^2}{r}, \quad (7.6)$$

which solves to

$$V_{orb}(r) = \sqrt{\frac{GM}{r}}. \quad (7.7)$$
Escape speed

\[ W = \int_R^\infty \frac{GMm}{r^2} dr = \frac{GMm}{R} \]

7.2. Surface escape speed \( V_{esc} \)

Another measure of the strength of a gravitational field is through the surface escape speed,

\[ V_{esc} = \sqrt{\frac{2GM}{R}}. \]  \hfill (7.4)

A object of mass \( m \) launched with this speed has a kinetic energy \( mV_{esc}^2/2 = GMm/R \). This just equals the work needed to lift that object from the surface radius \( R \) to escape at a large
7.4. Virial Theorem for bound orbits

If we define the gravitational energy to be zero far from a star, then for an object of mass $m$ at a radius $r$ from a star of mass $M$, we can write the gravitational binding energy $U$ as the negative of the escape energy,

$$U(r) = -\frac{GMm}{r}.$$  \hfill (7.8)

If this same object is in orbit at this radius $r$, then the kinetic energy of the orbit is

$$T(r) = \frac{m V_{orb}^2}{2} = +\frac{GMm}{2r} = -\frac{U(r)}{2},$$ \hfill (7.9)

where the second equation uses eqn. (7.7) for the orbital speed $V_{orb}(r)$. We can then write the total energy as

$$E(r) \equiv T(r) + U(r) = -T(r) = \frac{U(r)}{2}.$$  \hfill (7.10)

This fact that the total energy $E$ just equals half the gravitational binding energy $U$ is an example of what is known as the Virial Theorem. It is applicable broadly to most any stably
7.5. Questions and Exercises

Quick Question 1: In CGS units, the sun has log $g_{\odot} \approx 4.44$. Compute the log $g$ for stars with:
   a. $M = 10M_\odot$ and $R = 10R_\odot$
   b. $M = 1M_\odot$ and $R = 100R_\odot$
   c. $M = 1M_\odot$ and $R = 0.01R_\odot$

Quick Question 2:
   The sun has an escape speed of $V_{e\odot} = 618$ km/s. Compute the escape speed $V_e$ of the stars in parts a-c of QQ1.

Quick Question 3:
   The earth has an orbital speed of $V_e = 2\pi$ au/yr = 30 km/s. Compute the orbital speed $V_{orb}$ (in km/s) of a body at the following distances from the stars with the quoted masses:
   a. $M = 10M_\odot$ and $d = 10$ au.
   b. $M = 1M_\odot$ and $d = 100$ au.
   c. $M = 1M_\odot$ and $d = 0.01$ au.
In cgs units, the log of sun’s gravity is \( \log g_{\text{sun}} = 4.4 \). What is \( \log g \) for a star with mass \( M = 10 \, M_{\text{sun}} \) and radius \( R = 10 \, R_{\text{sun}} \)?

A. 4.4
B. 5.4
C. 3.9
D. 3.4
E. Not enough information to answer.
Clicker question

The sun has an escape speed $V_{e,s} = 620 \text{ km/s}$. What is the escape speed from a star with mass $M = 10 \ M_{\text{sun}}$ and radius $R = 10 \ R_{\text{sun}}$?

A. 62 km/s
B. 620 km/s
C. 6200 km/s
D. 200 km/s
E. 2000 km/s
Clicker question

The earth at 1 au from the sun has an orbital speed $V_e = 30 \text{ km/s}$. What is the orbital speed of a planet orbiting a star of mass $M = 10 M_{\text{sun}}$ at a distance of $d = 0.1 \text{ au}$?

A. 3 km/s
B. 30 km/s
C. 300 km/s
D. 90 km/s
E. 900 km/s
Inferring stellar motion

• Across the sky
  – “Proper motion”

• Toward/away from us
  – “Radial velocity”
  – Doppler shift
Proper Motion & Tranverse Speed

\[ V_t = \frac{\mu}{p} \text{ au/yr} = 4.7 \frac{\mu}{p} \text{ km/s}. \]
\[ F_g = \frac{GM_1M_2}{a^2}, \]  
\[ \text{Equation 10.1} \]

A key difference from the case of a satellite orbiting the earth, or a planet orbiting a star, is that in binary stars, the masses can become comparable. In this case, each star (1,2) now moves around the center of mass at a fixed distance \( a_1 \) and \( a_2 \), with their ratio given by \( a_2/a_1 = M_1/M_2 \) and their sum by \( a_1 + a_2 = a \). In terms of the full separation, the orbital distance of, say, star 1 is thus given by

\[ a_1 = a \frac{M_2}{M_1 + M_2}. \]  
\[ \text{Equation 10.2} \]

For the given period \( P \), the associated orbital speeds for star 1 is is given by \( V_1 = 2\pi a_1/P \). For a stable, circular orbit, the outward centrifugal force on star 1,

\[ F_{c1} = \frac{M_1V_1^2}{a_1} = \frac{4\pi^2 M_1 a_1}{P^2} = \frac{4\pi^2 a}{P^2} \frac{M_1 M_2}{M_1 + M_2}, \]  
\[ \text{Equation 10.3} \]

must balance the gravitational force from eqn. (10.1), yielding

\[ \frac{GM_1 M_2}{a^2} = \frac{4\pi^2 a}{P^2} \frac{M_1 M_2}{M_1 + M_2}. \]  
\[ \text{Equation 10.4} \]

This can be used to obtain the sum of the masses,

\[ M_1 + M_2 = \frac{4\pi^2 a^3}{G P^2} = \frac{a_{2u}^3}{P^2 y^3} M_{\odot}, \]  
\[ \text{Equation 10.5} \]
\[ F_g = \frac{GM_1M_2}{a^2}. \]  

(10.1)

A key difference from the case of a satellite orbiting the earth, or a planet orbiting a star, is that in binary stars, the masses can become comparable. In this case, each star (1,2) now moves around the *center of mass* at a fixed distance \( a_1 \) and \( a_2 \), with their ratio given by \( \frac{a_2}{a_1} = \frac{M_1}{M_2} \) and their sum by \( a_1 + a_2 = a \). In terms of the full separation, the orbital distance of, say, star 1 is thus given by

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(10.3)

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\[ \frac{GM_1M_2}{a^2} = \frac{4\pi^2 a}{P^2} \frac{M_1 M_2}{M_1 + M_2}. \]  

(10.4)

This can be used to obtain the sum of the masses,

\[ M_1 + M_2 = \frac{4\pi^2}{G} \frac{a^3}{P^2} = \frac{a_{au}^3}{P_{yr}^2} M_\odot, \]  

(10.5)
$\frac{M_1}{M_2} = 3.6; \; e = 0.0$
M1/M2 = 3.6; e = 0.4
Doppler shift and Radial velocity

$$z \equiv \frac{\Delta \lambda}{\lambda_o} = \frac{V_r}{c}$$

Lower Frequency

Higher Frequency
Hot Gas

Cold Gas

Continuum Spectrum

Emission Line Spectrum

Absorption Line Spectrum

Star

Photosphere: “Continuum Source”

Outer layers are Cooler -- Absorb Photons

See this
observer

Observed Spectrum
We see light from both A and B.

We see light from all of B, some of A.

We see light from both A and B.

We see light only from A.