Radiation vs. Gas Pressure,
the Stellar Mass-Luminosity Relation,
and the Eddington Limit

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1. Introduction

Spurred by ongoing discussions in our group, as well as by prearation for my upcoming
talks in Canada and Israel, I've been thinking more about the underlying physical basis for
the increase in stellar luminosity with mass, and how this is affected by the approach to the
Eddington limit. The notes here are an attempt to lay this out in as simple and straightforward
way possible, building in some ways on the basis developed last winter in Rich, Ken and my
analyses of stellar structure within the “Eddington Standard Model” (ESM).¹

Compared to those notes, the analysis here is quite rough and approximate, but hope-
fully it will help further clarify some key physical issues.

I begin in §2 with a brief recap of how the relative importance of radiative vs. gas
pressure depends directly on the Eddington parameter Γ. §3 then uses simple two-point
(center-to-surface) forms of the hydrostatic equilibrium and radiative diffusion equations to
obtain the basic scalings of luminosity with mass. §4 concludes with a summary discussion
of the overall significance of the results, along with a brief mention of the need for follow-on
work to understand better the various kinds of instabilities that might take hold near the
Eddington limit, leading to mass loss and an effective upper on a single star’s mass.

¹The present notes are intended to be self-contained. However, for background and reference, I’ve also
made the previous notes available in the web directory: http://www.bartol.udel.edu/~owocki/xfr/
See the files beginning with with “ESM” and then the author’s first name and approximate composition date in
the tile.
2. Dominance of Radiation Pressure near the Eddington Limit

In general the total pressure within a star can be written as the sum of the radiation and gas pressures, $P_{\text{tot}} = P_{\text{rad}} + P_{\text{gas}}$. The stratification of this total pressure with radius $r$ due to a local gravitational acceleration $g$ can be written in terms of the equation of hydrostatic equilibrium,

$$\frac{1}{\rho} \frac{dP_{\text{tot}}}{dr} = -g,$$

where $\rho$ is the mass density. For a medium with opacity $\kappa$, the gradient of the radiation pressure with radial optical depth, $d\tau = -\kappa \rho dr$, leads to a net radiative energy flux, as described by the radiative diffusion equation,

$$\frac{dP_{\text{rad}}}{d\tau} = \frac{F}{c}.$$

Noting then the common dependence of the two types of gradient on the density times radius, $\rho dr$, let us also cast the above equation of hydrostatic equilibrium in a similar, optical-depth form,

$$\frac{dP_{\text{tot}}}{d\tau} = \frac{F_c}{c},$$

where $F_c \equiv gc/\kappa$ is the critical flux for the Eddington limit condition at which the radiative acceleration equals gravity, $\kappa F_c/c = g$. Together these two equations imply that in the optically thick ($\tau \gg 1$) stellar interior, well below the surface boundary, the ratio of radiation to total pressure is just given by the Eddington parameter $\Gamma$,

$$\frac{P_{\text{rad}}}{P_{\text{tot}}} = \frac{F}{F_c} \equiv \Gamma ; \quad \tau \gg 1.$$

Since $P_{\text{tot}} = P_{\text{rad}} + P_{\text{gas}}$, this further implies that the ratio of gas pressure to total pressure is given by

$$\frac{P_{\text{gas}}}{P_{\text{tot}}} = 1 - \Gamma.$$

We thus see that in stars very near the Eddington limit, $\Gamma \rightarrow 1$, the overall pressure is dominated by radiation, with gas pressure playing only a vanishingly small role.
3. Mass-Luminosity Scaling Law

3.1. The Sub-Eddington Limit

We can also use these two basic equations of stellar structure – viz. hydrostatic equilibrium and radiative diffusion – to derive the basic scaling laws for the variation of stellar luminosity with mass.

For simplicity, let us first consider cases well below the Eddington limit, $\Gamma \ll 1$, at which the pressure is dominated the ordinary gas component, and thus given by the ideal gas relation in terms of the density $\rho$ and temperature $T$. Using then a Newtonian scaling for the gravitational acceleration, $g \sim M/R^2$, in terms of the stellar mass $M$ and radius $R$, we can apply a simple two-point scaling form for the hydrostatic equilibrium to derive a characteristic temperature for the stellar interior,

$$\frac{\rho T}{R} \sim \rho \frac{M}{R^2} \rightarrow T \sim \frac{M}{R}.$$  \hfill (6)

This can also be thought of as a specific example of the general Virial relation between the internal thermal energy ($\sim T$) and the gravitational binding energy ($\sim M/R$).

Similarly, using the temperature scaling of the radiation pressure, $P_{\text{rad}} \sim T^4$, and defining the radiative flux $F \sim L/R^2$ in terms of the stellar luminosity $L$ and stellar radius $R$, we can write a similar two-point form for the radiative diffusion equation,

$$\frac{P_{\text{rad}}}{\kappa \rho R} \sim \frac{L}{R^2},$$  \hfill (7)

where we have written the total optical depth $\tau \sim \kappa \rho R$ in terms of a mean stellar density $\rho$. Since this density scales with mass and radius as $\rho \sim M/R^3$, we find that the luminosity scales as

$$L \sim \frac{T^4 R^4}{\kappa M}.$$  \hfill (8)

Finally, if we then apply the temperature scaling from the hydrostatic equilibrium condition, we obtain

$$L \sim \frac{M^3}{\kappa}.$$  \hfill (9)

Even if we ignore the modest trend for opacity to decrease with mass (due to the hotter, more ionized envelope), we thus see that the luminosity of sub-Eddington stars tends to increase rather steeply with stellar mass, i.e. as $L \sim M^3$.

A particularly remarkable feature of this scaling is that the stellar radius completely cancels out. As emphasized in Rich’s notes from last winter, this is also a common feature
of detailed stellar structure models, particularly in the “Eddington Standard Model” (ESM) case for which \( \rho \sim T^3 \), implying then a direct proportionality between gas and radiation pressure, and thus giving the stellar structure a radiation-dominated character even for cases well below the Eddington limit. Overall, the cancellation of the radius here means that the stellar radius must be determined from additional physical contraints beyond just the hydrostatic and radiative diffusion equations.

3.2. Mass-Luminosity Scaling Incorporating the Eddinton Limit

Eddington first noted how this tendency for luminosity to increase with a high power of the stellar mass would imply a key, limiting role for radiation pressure in the structure of higher mass stars. This is because the ratio of radiative force to gravity scales as

\[
\Gamma = \frac{\kappa F/c}{g} = \frac{\kappa L}{4\pi GMc} \sim \frac{L}{M},
\]

which together with the above scaling of \( L \sim M^3 \) gives

\[
\Gamma \sim M^2,
\]

implying that high mass stars must become near the Eddington limit, \( \Gamma = 1 \).

Of course, the above scaling is based on the assumption that the pressure is dominated by the gas component, which according to the analysis in §2 becomes invalid if a star is near the Eddington limit. However, a simple way to incorporate the effects of radiation pressure is through the associated radiative-force reduction of the effective gravity by a factor \( 1 - \Gamma \). Thus by simply defining an effective gravitational mass as \( M \rightarrow M_{\text{eff}} \equiv M(1 - \Gamma) \), we can retain the above description of the hydrostatic stratification of just the gas pressure. The analysis then follows through as before, except that now this extra correction gives the final scaling the form,

\[
\Gamma = \Gamma_\odot \left( \frac{M}{M_\odot} \right)^2 (1 - \Gamma)^4,
\]

where we’ve now normalized in terms of the solar value for the Eddington parameter, which is quite small (\( \Gamma_\odot \approx 2.5 \times 10^{-5} \ll 1 \)), and the solar mass \( M_\odot \). Since this is a quartic equation in \( \Gamma \), explicit solutions are complicated, but it is trivial to solve the equation numerically.

Fig. 1 shows log-log plots of the resulting variation of \( \Gamma \) vs. \( M \), with the black curve showing the full solution from eqn. (12). For comparison, the red curve plots the simple power relation \( \Gamma \sim M^2 \) of eqn. (11), and the blue curve shows an approximate solution
Fig. 1.— Log-log plot of Eddington parameter $\Gamma$ vs. mass in solar units $M/M_\odot$, for the full solution of eqn. (12) (black), the analytically explicit approximation of eqn. (13) (blue), and the sub-Eddington limit scaling of eqn. (11) (red).
wherein the 4th power exponent on $1 - \Gamma$ has been replaced with unity, which thus allows the simple explicit solution,

$$\Gamma \approx \frac{1}{(M_l/M)^2 + 1},$$

where $M_l \equiv M_\odot / \sqrt{\Gamma_\odot}$ represents a characteristic mass for nearing the Eddington limit.

The plots in fig. 1 use a pure-electron opacity for the solar Eddington parameter, $\Gamma_\odot \approx 2.5 \times 10^{-5}$; but if one accounts for the role of bound-free opacity in the relatively lower temperature solar interior, then in principle this could be increased by a factor of a few (perhaps up to ten?), with an associated increase in the overall level of the positive slope part of the curves in fig. 1, by a factor proportion to the square-root of this opacity increase. This thus also implies a similar factor decrease in the limiting mass $M_l$.

### 3.3. Comparison with Observationally Inferred Masses and Scaling Laws

Overall, we see that this limiting mass is on the order of $M_l \approx 100 - 200 M_\odot$, and characterizes the mass at which stars approach the Eddington limit.

It is interesting to note that this is quite comparable to the observed upper limit of stellar masses in the galaxy. The record for direct mass measurement from binary orbit is ca. 80 $M_\odot$, and spectroscopic estimates for the most luminous stars in the galaxy, for example $\eta$ Carinae and the Pistol star, are of order 100-150 $M_\odot$.

More broadly, masses inferred for a broad range of stars in binaries also generally confirms a steep dependence of luminosity on mass for stars below the Eddington limit, with sometimes even a slightly higher exponent than the 3 derived here. As noted above, even that extra power can be obtained in this simple derivation if one accounts for a slight inverse mass dependence of the characteristic opacity, e.g. $\kappa \sim M^{-1/2}$. But in reality, the reason for the detailed differences are likely to be more complex, e.g. associated with the convective nature of stellar envelopes at lower mass.

Nonetheless, it is remarkable how well the overall scalings can be reproduced with just a simple analysis of the two key governing equations for hydrostatic equilibrium and radiative diffusion.

Note in particular that the results do not depend on any details of the actual energy-generation mechanism for the star! In my eyes, this is really quite remarkable, and it is the key reason that giants like Eddington and Schwarzschild were able to infer so much about the nature of stars even before the development of the modern understanding of nuclear energy generation (e.g. by Burbidge, Burbidge, Fowler and Hoyle).
4. Bottom Line and Future Directions

To me, there are two key conclusions from the above:

1. Simple scalings from stellar structure imply that stars near or above 100 $M_\odot$ must inevitably be near the Eddington limit, with radiation pressure dominating over gas pressure in the stellar envelope.

2. However, the Eddington limit itself acts as a kind of “barrier”, so static stars never cross it, implying that, in principle, there could be stars of arbitrarily large mass, with luminosity increasing in direct proportion to the mass, and the stars becoming arbitrarily close to the limit.

In practice, however, we only observe stars with masses of order 100 $M_\odot$, which seems to suggest strongly that the physics associated with the Eddington limit plays a key role in enforcing a real, practical upper limit on the masses of stars.

The specific link seems most likely to be related to instabilities that can occur near the Eddington limit, perhaps similar to the “photon bubbles” first suggested by Ed Spiegel, or more complex instabilities related to “Strange mode” pulsations, as analyzed by Wolfgang Glatzel, and most recently by Nir. A key factor in these instabilities is the fact that radiation pressure dominates over gas pressure, perhaps allowing radiative compression of the gas into dense clumps.

If so, this would lead then to a “porosity” reduction in the effective opacity, which could then allow a sudden release of pent-up energy in the stellar envelope, and thus episodes of super-Eddington luminosity. During such periods, the porosity reduction in radiative force in the dense interior would still allow the deep stellar envelope to remain gravitationally bound. But near the surface, where the overall lower density makes the clumps become optically thin, and thus exposed to the full radiative force, the material will be driven away, leading to a strong stellar wind. The continuum nature of the driving, moreover, can allow a much higher mass loss than from the usual line-driven stellar wind. Indeed, such continuum-driven mass loss rates could even approach the fundamental energy, or “photon tiring” limit, much as inferred for the giant eruption of Eta Carinae.

All this is based on what kind of instabilities might ensue near the Eddington limit.

Overall, then it seems a key initial focus of our newly funded NSF project should be to understand better the nature of such instabilities. In preparation for my visit with Nir in Israel at the end of October, I thus plan to focus more directly on this issue, with perhaps further additions to these notes. So stay tuned...