Homework: Due 11-17-04

This set (#9) is due on Wednesday, 11/17/2004.

- Sakurai Chap 3: 26, 27, 28. 4.2, 4.4 Merzbacher 17.6

- Merzbacher 17.6

The effective gyromagnetic ratio $g$ is defined by $\mu = \frac{e}{2mc} g I$, where $\mu$ is the magnetic moment operator and $I$ is the total spin of a nucleus. In a typical application, an external field may define a preferred direction, which is taken to be the $z$-direction, and one writes $\mu_z = \frac{e}{2mc} g I_z$. The spin of a nucleus is just the total angular momentum of the components, i.e. $I \equiv J = \sum_j J_i = \sum_j L_i + S_i$. In the nuclear shell model, the contributions from a full shell of nucleons is typically $I_{\text{shell}} = 0$. In a nucleus where only a single nucleon lies outside a full shell, the magnetic moment is dominated by that one nucleon, in which case $\mu_z = \frac{e}{2mc} \left( g_L L_z + g_S S_z \right)$. For a nucleon in a particular state $|jm\rangle$, the gyromagnetic ratio is given by

$$g = \frac{\langle jm| g_L L_z + g_S S_z |jm\rangle}{\langle jm| J_z |jm\rangle}$$  \hspace{1cm} (1)$$

At first it may appear that $g$ is a function of both $j$ and $m$, but since $J$, $L$, and $S$ are all vector operators one may use the Wigner-Eckart theorem to show there is no $m$-dependence. For example, rewriting $J$ as a spherical tensor of rank-1, $J_z = J^0_1$, and

$$\langle jm| J_z |jm\rangle = \langle jm| J^0_1 |jm\rangle = c^{1j}_{jm0m} \langle j|| J || j \rangle$$

Similarly

$$\langle jm| L_z |jm\rangle = \langle jm| L^0_1 |jm\rangle = c^{1j}_{jm0m} \langle j|| L || j \rangle$$

$$\langle jm| S_z |jm\rangle = \langle jm| S^0_1 |jm\rangle = c^{1j}_{jm0m} \langle j|| S || j \rangle$$

in which case the CG coefficients are common, and

$$g = \frac{g_L \langle j|| L || j \rangle + g_S \langle j|| S || j \rangle}{\langle j|| J || j \rangle}$$

and is independent of $m$. It remains to calculate $g$. This could be done by finding the reduced matrix elements, or it could be done by calculating the ratio in eq. (1) directly for the state $|jj\rangle$. For example, in the $j = l + \frac{1}{2}$ representation, $|jj\rangle = |ll\rangle \otimes |\frac{1}{2} \frac{1}{2}\rangle$, so

$$j g_+ = j \frac{\langle jj| g_L L_z + g_S S_z |jj\rangle}{\langle jj| J_z |jj\rangle} = j g_L \langle jj| L_z (|ll\rangle \otimes |\frac{1}{2} \frac{1}{2}\rangle) + g_S \langle jj| S_z (|ll\rangle \otimes |\frac{1}{2} \frac{1}{2}\rangle)$$

$$= g_L (j j) L_z (|ll\rangle \otimes |\frac{1}{2} \frac{1}{2}\rangle) + g_S (j j) S_z (|ll\rangle \otimes |\frac{1}{2} \frac{1}{2}\rangle)$$

$$= lg_L + \frac{1}{2} g_S = (j - \frac{1}{2}) g_L + \frac{1}{2} g_S$$

and in the $j = l - \frac{1}{2}$ representation

$$|jj\rangle = \sqrt{\frac{1}{2l+1}} |l, l - 1\rangle \otimes |\frac{1}{2} \frac{1}{2}\rangle - \sqrt{\frac{2l}{2l+1}} |l\rangle \otimes |\frac{1}{2} -\frac{1}{2}\rangle$$

...
so

\[ jg^- = g_L \langle j | j \rangle L_c \left( \sqrt{\frac{1}{2l+1}} | l, l - 1 \rangle \otimes | \frac{1}{2}, \frac{1}{2} \rangle - \sqrt{\frac{2l}{2l+1}} | l \rangle \otimes | \frac{1}{2}, \frac{-1}{2} \rangle \right) + \]

\[ \hspace{1cm} g_S \langle j | j \rangle S_c \left( \sqrt{\frac{1}{2l+1}} | l, l - 1 \rangle \otimes | \frac{1}{2}, \frac{1}{2} \rangle - \sqrt{\frac{2l}{2l+1}} | l \rangle \otimes | \frac{1}{2}, \frac{-1}{2} \rangle \right) \]

\[ = g_L \langle j | j \rangle \left( (l - 1) \sqrt{\frac{1}{2l+1}} | l, l - 1 \rangle \otimes | \frac{1}{2}, \frac{1}{2} \rangle - l \sqrt{\frac{2l}{2l+1}} | l \rangle \otimes | \frac{1}{2}, \frac{-1}{2} \rangle \right) + \]

\[ \hspace{1cm} g_S \langle j | j \rangle \left( \frac{1}{2} \sqrt{\frac{1}{2l+1}} | l, l - 1 \rangle \otimes | \frac{1}{2}, \frac{1}{2} \rangle + \frac{1}{2} \sqrt{\frac{2l}{2l+1}} | l \rangle \otimes | \frac{1}{2}, \frac{-1}{2} \rangle \right) \]

\[ = g_L((l - 1) \frac{1}{2l+1} + l \frac{2l}{2l+1}) + g_S(\frac{1}{2} \frac{1}{2l+1} - \frac{1}{2} \frac{2l}{2l+1}) \]

\[ = g_L \left( \frac{2l + l - 1}{2l+1} \right) - \frac{g_S}{2} \left( \frac{2l - 1}{2l+1} \right) = \frac{l}{j+1} ((l + 1) g_L - \frac{1}{2} g_S) \]

\[ = \frac{l}{j+1} \left( (j + \frac{3}{2}) g_L - \frac{1}{2} g_S \right) \]

After the algebra, one can make a plot, for neutrons and protons. Color coding is (black, red, blue, green) → (g⁺, g⁻, g⁺⁻, g⁻⁻)

![Graph](image)

- **Sakurai 3.26**

a) Adopt the notation \( U = (U_1, U_2, U_3), V = (V_1, V_2, V_3) \) and write the tensors as \( T = T_{ij} U_i V_j \). Specifying the components of the tensor is equivalent to writing \( T \) out completely. A scalar is then \( S = S_{ij} U_i V_j \), with

\[ S_{ij} = \delta_{ij} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \]  

To make a cartesian vector tensor one would proceed with \( W = U \times V \) or \( W_k = i \epsilon_{ijk} U_i V_j \). A spherical rank-1 tensor can built by

\[ W_+ = \frac{1}{\sqrt{2}} (W_1 + iW_2) \]

\[ W_0 = W_3 \]

\[ W_- = \frac{1}{\sqrt{2}} (W_1 - iW_2) \]

Alternatively, to build higher rank tensors, one can first make spherical tensors for \( U \) and \( V \), and then combine these using the rules for angular momentum addition. For example,
Similarly the spherical rank-1 tensor is given by $T^1_q = \sum c_{i1q,q'=q-1} U_q V_{q-1}$.

\[ T_0^1 = \sum c_{i1q,1-q} U_q V_{1-q} = \frac{1}{\sqrt{2}} (U+ V_0 - U_0 V_+) = \frac{1}{\sqrt{2}} ((U_1 + iU_2) V_3 - U_3(V_1 + iV_2)) = \frac{1}{\sqrt{2}} ((U_1 V_3 - U_3 V_1) + i(U_2 V_3 - U_3 V_2)) = \frac{1}{\sqrt{2}} (W_1 + iW_2) = \frac{1}{\sqrt{2}} W_+ \]

\[ T_0^0 = \sum c_{i0q,q} U_q V_{q-1} = \frac{1}{\sqrt{3}} (U_+ V_- - U_0 V_0 + U_- V_+) = -\frac{1}{\sqrt{3}} ((U_1 + iU_2) V_3 - U_3(V_1 + iV_2)) - \frac{1}{\sqrt{3}} ((U_1 - iU_2) (V_1 + iV_2)) = -\frac{1}{\sqrt{3}} ((U_1 V_3 - U_3 V_1) + i(U_2 V_3 - U_3 V_2)) = -\frac{1}{\sqrt{3}} (U_1 V_1 + U_2 V_2 + U_3 V_3) = \frac{1}{\sqrt{3}} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \]

which again, apart from the normalization, is the same as the cartesian construction. The coefficient matrices are

\[ T_1^1 = \frac{1}{2} \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & i \\ 1 & i & 0 \end{array} \right), \quad T_0^1 = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -i \end{array} \right), \quad T^1_1 = \frac{1}{2} \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{array} \right) \]

For the rank-2 tensor, $T^2_q = \sum c_{i2q,q'} U_q V_{q'-q}$.
$T_2^2 = \Sigma_q \frac{c_{11}^{11} q_{2,q-2} U_q V_{2-q}}{\sqrt{2}} U_+ V_+ = \frac{1}{2} ((U_1 + i U_2) (V_1 + i V_2)) = \frac{1}{2} (U_1 V_1 - U_2 V_2 + i(U_2 V_1 + U_1 V_2))$  

$T_1^2 = \Sigma_q \frac{c_{11}^{11} q_{1,q-1} U_q V_{1-q}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} (U_+ V_0 + U_0 V_+ - \frac{1}{2} ((U_1 + i U_2) V_3 + U_3 (V_1 + i V_2)) = -\frac{1}{2} ((U_1 V_3 + U_3 V_1) + i(U_2 V_3 + U_3 V_2))$  

$T_0^2 = \Sigma_q \frac{c_{11}^{11} q_{0,q-0} U_q V_{0-q}}{\sqrt{2}} = \frac{1}{\sqrt{6}} (U_+ V_+ + 2 U_0 V_0 + U_0 V_+ - \frac{1}{\sqrt{6}} (U_1 + i U_2) (V_1 - i V_2) + 2 U_0 V_0 - \frac{1}{\sqrt{6}} (U_1 - i U_2) (V_1 + i V_2))$  

$T_{-1}^2 = \Sigma_q \frac{c_{11}^{11} q_{1,q-1} U_q V_{-1-q}}{\sqrt{2}} = \frac{1}{\sqrt{2}} (U_+ V_0 + U_0 V_- - \frac{1}{\sqrt{2}} ((U_1 - i U_2) V_3 + U_3 (V_1 - i V_2)) = \frac{1}{\sqrt{2}} ((U_1 V_3 + U_3 V_1) - i(U_2 V_3 + U_3 V_2))$  

$T_{-2}^2 = \Sigma_q \frac{c_{11}^{11} q_{2,q-2} U_q V_{-2-q}}{\sqrt{2}} = U_- V_- = \frac{1}{2} ((U_1 - i U_2) (V_1 - i V_2)) = \frac{1}{2} (U_1 V_1 - U_2 V_2 - i(U_2 V_1 + U_1 V_2))$ 

The coefficient matrices are

$T_2^2 = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{6}} & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_1^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}, \quad T_0^2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad T_{-1}^2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}, \quad T_{-2}^2 = \frac{1}{2} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

It is worth noting some properties of the $T$ matrices. First, except for $T_0^k$, all $T_q^k$ are traceless, $\Sigma_i (T_q^k)^* = 0$. Second, by inspection, the $T_q^k$ are all orthogonal in the following sense, $\Sigma_i (T_q^k)^* (T_q^k)^{ij} = \delta_{kk'} \delta_{qq'}$ (note: this is not normal matrix multiplication). Third, $T^1$ is antisymmetric: $(T_q^1)_{ij} = -(T_q^1)_{ji}$, while $T^2$ is symmetric: $(T_q^2)_{ij} = (T_q^2)_{ji}$. Fourth, a rotation operating on $T^1$ still leaves a component of $T^1$ orthogonal to $T^0$ and $T^2$. To see this, the effect of a rotation is

$(T_q^1)_{ij} \rightarrow (T_q^1)_{ij} = R_{km} R_{jn} (T_q^1)_{mn}$

so, the sum over components can be seen to vanish by using the antisymmetric and symmetric properties of the tensors.

$\Sigma_i (T_q^1)^* (T_q^2)_{ij} = \Sigma_i (T_q^1)^* (T_q^1)_{mn} (T_q^2)_{ij} = \Sigma_i (T_q^1)^* (T_q^1)_{mn} (T_q^2)_{ij} = \Sigma_i (T_q^1)^* (T_q^1)_{mn} (T_q^2)_{ij}$

This is an explicit demonstration that the $k = 1$ and $k = 2$ representations don't mix. Alternatively, under a rotation $T_q^k$ transforms as

$T_q^k \rightarrow (T_q^k)^* = \Sigma_q^{\prime} \mathcal{D}_q^{k} T_q^k$
so two representations remain orthogonal.

\[(T_{q_1}^{k_1})^* (T_{q_2}^k) = (T_{q_1}^{k_1})^* \sum_{q} D_{qq}^k T_{q_2}^k\]

\[= \sum_{q} D_{qq}^k \sum_{ij} (T_{q_1}^{k_1})^* (T_{q_2}^k)_{ij}\]

\[= \sum_{q} D_{qq}^k \delta_k \delta_{q_1, q}\]

- **Sakurai 3.27**

- **a) Relate various matrix elements using the Wigner-Eckart theorem**

\[\langle n' l' m' \rangle - \frac{1}{\sqrt{2}} (x + i y) |n l m\rangle,\, \langle n' l' m' \rangle |z| n l m\rangle,\, \langle n' l' m' \rangle \frac{1}{\sqrt{2}} (x - i y) |n l m\rangle\]

The operators \(-\frac{1}{\sqrt{2}} (x + i y),\, z,\, \frac{1}{\sqrt{2}} (x - i y)\) are components of a vector tensor operator \(T_q^1\) with \(q = 1, 0, -1\); c.f. 3.10.16 of Sakurai. Instead of using spherical harmonics, one can use the definition of a tensor operator, 3.10.25

\[\begin{align*}
[J_z, T_{q}^{k}] &= q T_{q}^{k} \\
[J_+, T_{q}^{k}] &= \sqrt{(k + q)(k + q + 1)} T_{q \pm 1}^{k}
\end{align*}\]

For position states/operators the relevant generators are \(J \to L\). One can explicitly check that \(X_{\pm,0}\) defined by

\[X_0 = z\]

\[X_\pm = \mp \frac{1}{\sqrt{2}} (x \pm i y)\]

act as components of a \(k = 1\) tensor operator. For example,

\[\begin{align*}
[L_z, X_0] &= 0 \\
[L_z, X_+] &= X_+ \\
[L_+, X_+] &= 0 \\
[L_+, X_-] &= \sqrt{2} X_0
\end{align*}\]

etc, can be checked by explicit calculation.

Having verified that \(X_{\pm,0}\) satisfy the definition of a tensor operator, one can use the Wigner-Eckart theorem to partially evaluate the matrix elements.

\[X_+ :\begin{align*}
\langle n' l' m' \rangle - \frac{1}{\sqrt{2}} (x + i y) |n l m\rangle &= \langle n' l' m' \rangle |T_q^1 |n l m\rangle = \langle l 1; l' m' | l 1; m 1\rangle A \\
X_0 :\begin{align*}
\langle n' l' m' \rangle |z| n l m\rangle &= \langle n' l' m' \rangle |T_0^1 |n l m\rangle = \langle l 1; l' m' | l 1; m 0\rangle A \\
X_- :\begin{align*}
\langle n' l' m' \rangle \frac{1}{\sqrt{2}} (x - i y) |n l m\rangle &= \langle n' l' m' \rangle |T_{-1}^1 |n l m\rangle = \langle l 1; l' m' | l 1; m; -1\rangle A
\end{align*}\]

where \(A = \frac{\langle n' l' m' | T_q^1 | n l m\rangle}{\sqrt{2j + 1}}\) is independent of \(q\).

There are selection rules: for \(T_q^1\) : \(m' = m + q\). There is a general angular momentum selection rule that for a \(T^1\) operator \(j - 1 \leq j' \leq j + 1\). Additionally, \(j = j' = 0\) is not allowed. In the present case there is no spin so \(j = l\), so \(l - 1 \leq l' \leq l + 1\). Since the operator \(x\) is odd under parity, and the angular momentum states have parity \(\pi_l = (-1)^l\), it is required that \(\Delta l = \text{odd}\). Combined with the angular momentum restriction \(l' = l \pm 1\).
One of the main applications of these selection rules is to radiative transitions, discussed later under perturbation theory. The leading term is often the "electric dipole" transition, which is of the form \( \langle f \mid x \mid i \rangle \), where \( i, f \) represent initial and final states. If the states are given in an \( |lsjm_i \rangle \) basis then both angular momentum and parity selection rules apply. However, if the states are given in \( |lsjm \rangle \) basis then only angular momentum rules apply, since the \( |jm \rangle \) are generally mixtures of different \( l \)-states and are not eigenstates of parity.

There is one more selection rule which bears mentioning. The CG coefficients obey the rule that under exchange of \( (m, q, m') \rightarrow (-m, -q, -m') \) the CG coefficients change by \((-)^{j+k+j'+2m'} \). A consequence of this is that for \( m = m' = q = 0 \) and \( j' = j = \) integer, electric dipole transitions are not allowed. I confess I do not know how to prove this in a simple fashion. For a discussion, see for example Landau & Lifshitz chapter 14, which develops 3-\( j \) symbols (equivalent to CG coefficients) from a discussion of spinors. I am not sure if the discussion in 3.8 of Sakurai is equivalent. It is introducing some concepts of spinors, but doesn't use them to study 3-\( j \) symbols. An exercise for the reader.

\( \text{b) Evaluated using wavefunctions, e.g.} \)

\[
\langle n' l' m' \mid z \mid n l m \rangle = \int d\hat{x} \, d\hat{x}' \, \langle n' l' m' \mid \hat{x}' \rangle \, \langle \hat{x} \mid z \rangle \, \langle \hat{x} \mid n l m \rangle = \int d\hat{x} \, d\hat{x}' \, \langle n' l' m' \mid \hat{x}' \rangle \, z \, \delta(\hat{x}' - \hat{x}) \, \langle \hat{x} \mid n l m \rangle = \int d\hat{x} \, \psi^*_{nlm}(\hat{x}) \, z \, \psi_{nlm}(\hat{x}) = \int r^2 \, d\Omega \, R_{lm}^*(\hat{r}) \, R_{l'm'}(\hat{r}) \, r \, \cos \theta \, R_{nl}(r) \, Y_{ln}(\hat{n}) = \int r^2 \, d\Omega \, R_{lm}^*(\hat{r}) \, R_{nl}(r) \, \int d\Omega \, Y_{l'm'}(\hat{n}) \, \cos \theta \, Y_{ln}(\hat{n}) = I_R \, I_\theta
\]

In general, it is rather annoying to have to calculate integrals for each case. It should be clear, however, that the angular integral is related to the Clebsch-Gordon coefficient in the Wigner-Eckart theorem. Specifically

\[
\cos \theta = \sqrt{\frac{4\pi}{3}} \, Y_{10}(\hat{n})
\]

and so we can use Sakurai 3.7.73 to evaluate the integral of the product of three spherical harmonics

\[
I_\theta = \sqrt{\frac{4\pi}{3}} \, \int d\Omega \, Y_{lm}^*(\hat{n}) \, Y_{ln}(\hat{n}) \, Y_{lm}(\hat{n}) \text{ which follows from the orthogonality of the } Y_{lm}.
\]

\[
= \sqrt{\frac{4\pi}{3}} \sqrt{\frac{(2l+1)(3)!}{4\pi(2l+1)!}} \langle l \, l' \, 0 \mid l \, l \, 0 \rangle \langle l \, l' \mid l \, l \rangle \langle m \mid m \rangle
\]

\[
= \sqrt{\frac{2l+1}{2l+1}} \langle l \, l' \mid l \, l \rangle \langle m \mid m \rangle
\]

Having associated the angular integral with the C-G coefficient, it follows that \( I_R \) is related to the reduced matrix element

On the one hand

\[
\langle n' l' m' \mid z \mid n l m \rangle = \langle l \, l' \mid l \, m \rangle \langle l \, l \mid m \rangle \langle m \mid 0 \rangle \sqrt{\frac{(2l+1)!}{2l+1}} \frac{(m')^l |l| f}{\sqrt{2l+1}}
\]

while on the other
\( \langle n' l' m' | z | n l m \rangle = I_R I_\theta = I_R \sqrt{\frac{2 l' + 1}{2 l + 1}} \langle l' 0 | l 1 ; 0 0 \rangle \langle l 1 ; l' m' | l 1 ; m 0 \rangle \)

Combining these results gives us the reduced matrix element

\( \langle n' l' || T^1 || n l \rangle = I_R \langle l 1 ; l' 0 | l 1 ; 0 0 \rangle \frac{2 l' + 1}{\sqrt{2 l + 1}} \)

### Sakurai 3.28

- **a)** write \( xy, xz, \) and \((x^2 - y^2)\) in terms of the components of a rank 2 spherical tensor.

Again, one can work directly from the spherical harmonics, which are identified as irreducible tensor operators.

\[
Y_{2,\pm 2} = \sqrt{\frac{15}{32 \pi}} \frac{(x \pm i y)^2}{r^2}
\]

\[
Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8 \pi}} \frac{z(x \pm i y)}{r^2}
\]

\[
Y_{2,0} = \sqrt{\frac{5}{16 \pi}} \frac{(3 z^2 - r^2)}{r^2}
\]

Instead, one can build up a \( k = 2 \) spherical tensor from \( X_{x,0} \). Specifically, using the theorem embodied in eq 3.10.27, tensor operators may be combined in a manner completely analogous to angular momentum addition. In the present case \( k_1 = k_2 = 1 \), and the tensor \( T^2 \) can be defined

\[ T^2_q = \sum_{q'} \langle 1 1 ; 2 q | 1 1 ; q' (q - q') \rangle X_{q'} X_{q - q'} \]

Suppressing the \( k_1 = k_2 = 1 \) simplifies the notation a little bit,

\[
T^2_2 = \langle 22 | 11 \rangle X_+ X_+ = \frac{1}{2} (x + i y)^2 = \frac{1}{2} (x^2 - y^2 + 2 i x y)
\]

\[
T^2_1 = \langle 21 | 10 \rangle X_+ X_0 + \langle 21 | 01 \rangle X_0 X_+ = -z(x + i y)
\]

\[
T^2_0 = \langle 20 | 1, -1 \rangle X_+ X_+ + \langle 20 | -1, 1 \rangle X_- X_+ + \langle 20 | 00 \rangle X_0 X_0 =
\]

\[
= \frac{1}{\sqrt{6}} (x + i y) (x - y) + \frac{1}{\sqrt{6}} (x + i y) (x - y) + \frac{z z}{\sqrt{6}}
\]

\[
= -\frac{1}{\sqrt{6}} (x + i y) (x - y) + \frac{z z}{\sqrt{6}} = \frac{1}{\sqrt{6}} (2 z z - x^2 - y^2) = \frac{1}{\sqrt{6}} (3 z^2 - r^2)
\]

\[
T^2_1 = \langle 2, -1 | 0, -1 \rangle X_0 X_+ + \langle 2, -1 | -1, 0 \rangle X_- X_0 = z(x - i y)
\]

\[
T^2_2 = \langle 2, -2 | -1, -1 \rangle X_- X_- = \frac{1}{2} (x - i y)^2 = \frac{1}{2} (x^2 - y^2 - 2 i x y)
\]

with these relations

\[
x y = -\frac{i}{2} (T^2_2 - T^2_{-2})
\]

\[
x z = -\frac{i}{2} (T^2_1 - T^2_{-1})
\]

\[
(x^2 - y^2) = T^2_0 + T^2_{-2}
\]

- **b)** Evaluate \((x^2 - y^2)\) matrix elements in terms of \( Q \)

\( Q \) is defined by
\[ Q = e^{\langle \alpha, j, j \rangle 3 z^2 - r^2 | \alpha, j, j \rangle} = \sqrt{6} e^{\langle \alpha, j, j | T_3^2 | \alpha, j, j \rangle} = \sqrt{6} e^{\langle j 2; j j | j 2; j 0 \rangle \frac{\langle \alpha, j | T_3^2 | \alpha, j \rangle}{\sqrt{2 j + 1}}} \]

where first the definition in part a) and then the Wigner-Eckart theorem have been used. The last relation can be rewritten as

\[ e^{\frac{\langle \alpha, j | T_3^2 | \alpha, j \rangle}{\sqrt{2 j + 1}}} = \frac{Q}{\sqrt{6} \langle j 2; j j | j 2; j 0 \rangle} \]

The requested matrix elements can now be evaluated in terms of \( Q \) by

\[ M = e^{\langle \alpha, j, m' | (x^2 - y^2) | \alpha, j, j \rangle} = e^{\langle \alpha, j, m' | T_2^2 + T_2^2 | \alpha, j, j \rangle} \]

by the result from part a). Since we are starting from the maximal \( m \)-state, the \( T_2^2 \) term does not contribute, so

\[ M = e^{\langle \alpha, j, m' | T_2^2 | \alpha, j, j \rangle} = \frac{Q}{\sqrt{6}} \frac{\langle j 2; j m' | j 2; j -2 \rangle \langle \alpha, j | T_2^2 | \alpha, j \rangle}{\langle j 2; j j | j 2; j 0 \rangle} \]

The Wigner-Eckart theorem allows matrix elements, expectation values, transition rates, etc. to be compared for similar processes. Experimentally, it is often easier to compare two rates, thus eliminating systematic uncertainties that may be common to both measurements, than it is to determine the absolute value of each measurement separately.

**Sakurai 4.2**

- \( T_d \quad T_d \quad [T_d, T_d] = 0 \quad [K_i, K_j] = 0 \quad \text{Generators of translations commute} \quad \text{Property of flat space} \)
- \( D_n \quad D_n' \quad [D_n, D_n'] = 0 \quad [J_i, J_j] = i \epsilon_{ijk} J_k \quad \text{Generators of SU}(2) \text{ do not commute} \quad \text{Rotation group is non-abelian} \)
- \( T_d \quad \Pi \quad [T_d, \Pi] = 0 \quad [K_i, \Pi] = 0 \quad \text{Momentum is parity odd operator} \quad \text{Reflection changes sign of reflection} \)
- \( D_n \quad \Pi \quad [D_n, \Pi] = 0 \quad [J_i, \Pi] = 0 \quad \text{Ang. momentum is parity even} \quad \text{True for} \ L = r \times p \quad \text{Defined for} \ S \)

**Sakurai 4.4**

a) The spin angular functions \( \mathcal{Y}_l^j \) are no more than the result of combining a representation \( l \) of orbital angular momentum with a spin-\( \frac{1}{2} \) representation. The rules of angular momentum addition apply

\[ \mathcal{Y}_l^j = | jm \rangle = \sum_{m} c^{1/2}_{jm, m, m_l | m_l m_s} | m_l \rangle \otimes | \frac{1}{2} m_s \rangle \]

where \( m_s = m - m_l \).

For the case \( l = 0 \), the \( \mathcal{Y} \) are particularly simple. \( j = 1/2 \) and \( m = m_s \)

\[ \mathcal{Y}_0^{1/2, 1/2} = | 00 \rangle \otimes | \frac{1}{2} \frac{1}{2} \rangle \]
b) It is possible to evaluate \( \sigma \cdot x \mathcal{Y}^{1/2,1/2} \) using the cartesian form for \( \sigma \cdot x \). Alternatively, one can write \( \sigma \cdot x \) in terms of spherical tensor operators \( \sigma \cdot x = \sum_q (-)^q \sigma^\dagger \cdot q \, x^\dagger_q \), where \( \sigma^\dagger_0 = \sigma_z \), \( \sigma^\dagger_\pm = \mp \frac{1}{\sqrt{2}} \, (\sigma_x \pm i \sigma_y) \), and similarly for \( x^\dagger \). In this form, \( \sigma^\dagger \) and \( x^\dagger \) are rank-1 tensor operators, and their action on \( \mathcal{Y}^{1/2,1/2} \) is dictated by the Wigner-Eckart theorem, e.g.

\[
\langle s'm_s' | \sigma^\dagger_q | s \rangle = \mathcal{C}_{q,m'_s,-q,m_s} \langle s' || \sigma^\dagger || s \rangle
\]

In the present case, the reduce matrix element vanishes unless \( s' = s \), since the spin operator doesn't change \( s \), but the relative strength of the different operators is still given by the relative Clebsch-Gordon coefficients. Accordingly,

\[
(\sigma \cdot x) \mathcal{Y}^{1/2,1/2} = \sum_q (-)^q x^\dagger_q \sigma^\dagger \cdot q (|00\rangle \otimes \frac{1}{2} \frac{1}{2})
\]

\[
= (-x^\dagger_0 \sigma^\dagger_0 + x^\dagger_0 \sigma^\dagger_0 - x^\dagger_1 \sigma^\dagger_1) (|00\rangle \otimes \frac{1}{2} \frac{1}{2})
\]

\[
= a_1 (|11\rangle \otimes \frac{1}{2} \frac{1}{2}) + a_0 (|10\rangle \otimes \frac{1}{2} \frac{1}{2})
\]

where \( a_1 = -\langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle \langle 1 || x^\dagger || 0 \rangle c_{10}^{1/2} c_{1/2,-1/2}^{1/2} = -\langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle \langle 1 || x^\dagger || 0 \rangle \cdot 1 \cdot (-\sqrt{2/3}) \), where the first CG coefficient is for the orbital angular momentum and the second is for the spin. Similarly,

\[
a_0 = \langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle \langle 1 || x^\dagger || 0 \rangle c_{10}^{1/2} c_{1/2,1/2}^{1/2} = \langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle \langle 1 || x^\dagger || 0 \rangle \cdot 1 \cdot 1/3 . \]

Combining the results

\[
(\sigma \cdot x) \mathcal{Y}^{1/2,1/2} = \langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle \langle 1 || x^\dagger || 0 \rangle \left( \sqrt{2/3} |1\rangle \otimes \frac{1}{2} \frac{1}{2} - \sqrt{1/3} |0\rangle \otimes \frac{1}{2} \frac{1}{2} \right) = \langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle \langle 1 || x^\dagger || 0 \rangle \mathcal{Y}^{1/2,1/2}
\]

For completeness, the reduced matrix elements can be determined by consideration of a particular convenient state. Specifically,

\[
\langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle = 1 = \langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle c_{10}^{1/2} c_{1/2,1/2}^{1/2} = \langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle \left( -\sqrt{1/3} \right) \Rightarrow \langle \frac{1}{2} || \sigma^\dagger || \frac{1}{2} \rangle = -\sqrt{3}
\]

\[
\langle 1 || x^\dagger || 0 \rangle = r \langle 1 || \cos \theta || 0 \rangle = \frac{r}{\sqrt{3}} = \langle 1 || x^\dagger || 0 \rangle c_{10}^{1/2} = \langle 1 || x^\dagger || 0 \rangle \Rightarrow \langle 1 || x^\dagger || 0 \rangle = \frac{r}{\sqrt{3}}
\]

so

\[
(\sigma \cdot x) \mathcal{Y}^{1/2,1/2} = -r \mathcal{Y}^{1/2,1/2}
\]

c) This result could have been anticipated on the basis of symmetry considerations. The operator \( \sigma \cdot x \) is a scalar (it transforms as \( T^0_0 \)) under rotations. It follows that it does not change the angular momentum state, i.e.

\[
\sigma \cdot x | jm \rangle = a | jm \rangle
\]

where \( a \) is a number. On the other hand, the operator is the product of two vector operators \( S \) and \( x \), and so has non-trivial behavior under \( S \) and \( L \). It follows that although \( j \) and \( m \) do not change, the value of \( l \) may change. Next consider that the action of a normalized tensor operator \( T^q_0 \) on a \( |00\rangle \) state yields the state \( |kq\rangle \). Since \( x \) is a vector operator \( T^1 \) under \( L \), it follows that the final orbital angular momentum is \( l = 1 \). Alternatively, one can consider parity. \( \sigma \cdot x \) is odd under parity. The state \( \mathcal{Y}^{1/2,1/2} \) has well defined \( l \), and so has well-defined parity \((-1)^l \). The action of an odd-parity operator on an even parity state results in an odd-parity state. So, the \( l = 1 \) nature of the result is consistent with parity.

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