Homework: Due 12-8-04

This set (#12) is due on Wednesday, 12/8/2004.

- **Sakurai 5.39**

The probability density in momentum space is given by

\[
\frac{d^3 p}{d^3 q} = |\phi(q)|^2
\]

where, for the ground state,

\[
\phi(q) = \frac{1}{(2\pi)^3} \int d^3 x e^{i q x} \psi_{100}(x) \\
= \frac{1}{(2\pi)^3} \int \frac{2}{\sqrt{4\pi}} \int r^2 d r d \phi e^{i q r} R_{10}(r) Y_0(\theta, \phi) \\
= \frac{1}{(2\pi)^3} \sqrt{\frac{2}{4\pi}} \int r^2 R_{10}(r) d r \left( \frac{e^{i q r} - e^{-i q r}}{i q r} \right)
\]

I've used \( q \) instead of \( p \), so I can reuse the notes from lecture, with minimal editing. Next, recall the ground state radial wave function

\[
R_{10}(r) = \frac{2}{a_0^2} e^{-r/a_0}
\]

so

\[
\phi(q) = \frac{1}{(2\pi)^3} \frac{2}{a_0^2} \sqrt{\frac{\pi}{q a_0^2}} \int r^2 d r e^{-r/a_0} \left( \frac{e^{i q r} - e^{-i q r}}{i q r} \right) \\
= -i \frac{1}{(2\pi)^3} \frac{2}{q a_0^2} \sqrt{\frac{\pi}{q a_0^2}} \int r d r e^{-r/a_0} (e^{i q r} - e^{-i q r}) \\
= -i \frac{1}{(2\pi)^3} \frac{2}{q a_0^2} \sqrt{\frac{\pi}{q a_0^2}} (z^2 - z_+^2)
\]

where \( \frac{1}{z_+} = \frac{1}{a_0} \pm i q \), or \( z_+ = \frac{a_0 (1 \pm i a_0 q)}{1 + a_0^2 q^2} \). Plugging in for \( z_+ \),

\[
|\phi(q)|^2 = \frac{1}{(2\pi)^6} \frac{4 \pi a_0^3}{q^3} \frac{|1 - i a_0 q|^2 - |1 + i a_0 q|^2|^2}{(1 + a_0^2 q^2)} \\
= \frac{1}{(2\pi)^6} \frac{4 \pi}{q^2 a_0} \frac{|-4 i a_0 q|^2}{(1 + a_0^2 q^2)} \\
= \frac{1}{(2\pi)^6} \frac{64 \pi a_0^3}{(1 + a_0^2 q^2)}
\]

- **Sakurai 5-40**

Calculate the lifetime for the \( 2 \ p \to 1 \ s \) transition of hydrogen. The lifetime is given by \( \tau = \frac{1}{\Gamma} \), so just calculate the total decay rate

\[
\Gamma = \sum \frac{\frac{1}{\lambda}}{(2\pi)^3} \int d^3 k \Gamma_{i \to f}
\]
The full specification of the unperturbed states includes the state of both the atom and the radiation field. The latter is specified by the number of photons. For the problem of atomic decay in a vacuum, the initial state has no photons, and the final state has one photon. The time dependent matrix element is

\[ V_{\ell f}(t) = \langle 1 s, k \lambda | - \frac{e}{m c} \cdot p \cdot A \cdot \mathbb{I} \cdot 2 \ p, 0 \rangle \]

\[ = \langle 1 s, k \lambda | - \frac{e}{m c} \cdot p \cdot \sum_{\lambda} \frac{L^2}{(2\pi)^2} \int d^3 k' N_{\omega} \left( a_{k' \lambda} e_{k' \lambda} e^{i(k' - k - \omega t)} + a_{k' \lambda}^\dagger e_{k' \lambda}^\dagger e^{-i(k' - k - \omega t)} \right) | 2 \ p, 0 \rangle \]

For this process, only creation operators contribute, so

\[ V_{\ell f}(t) = \langle 1 s, k \lambda | - \frac{e}{m c} \cdot p \cdot \sum_{\lambda} \frac{L^2}{(2\pi)^2} \int d^3 k' N_{\omega} \left( a_{k' \lambda}^\dagger e_{k' \lambda} e^{-i(k' - k - \omega t)} \right) | 2 \ p, 0 \rangle \]

At this point, the atomic and radiative matrix elements can be separated.

\[ V_{\ell f}(t) = - \frac{e}{m c} \sum_{\lambda} \frac{L^2}{(2\pi)^2} \int d^3 k' N_{\omega} \langle 1 s, k \lambda | p \cdot e_{k' \lambda}^\dagger a_{k' \lambda}^\dagger e^{-i(k' - k - \omega t)} | 2 \ p, 0 \rangle \]

\[ = - \frac{e}{m c} \sum_{\lambda} \frac{L^2}{(2\pi)^2} \int d^3 k' N_{\omega} \langle 1 s | p \cdot e_{k' \lambda}^\dagger e^{-i(k' - k) - \omega t} | 2 \ p \rangle e^{i\omega t} \langle k \lambda | a_{k' \lambda}^\dagger | 0 \rangle \]

\[ = - \frac{e}{m c} \sum_{\lambda} \frac{L^2}{(2\pi)^2} \int d^3 k' N_{\omega} \langle 1 s | p \cdot e_{k' \lambda}^\dagger e^{-i(k' - k) - \omega t} | 2 \ p \rangle e^{i\omega t} \left( \frac{2\pi}{L} \right)^3 \delta^3(k - k') \delta_{\lambda\lambda} \]

\[ = - \frac{e}{m c} N_{\omega} \langle 1 s | p \cdot e_{k \lambda}^\dagger e^{-i(k - k') - \omega t} | 2 \ p \rangle e^{i\omega t} \]

\[ = M_{\ell f} e^{i\omega t} \]

where \( M_{\ell f} \) is the transition matrix element, and the time dependence is in the factor of \( e^{i\omega t} \).

With this \( V(t) \), the rate \( \Gamma_{i\rightarrow f} \) is given by Fermi’s golden rule

\[ \Gamma_{i\rightarrow f} = \frac{2\pi}{\hbar} | M_{i\ell f} |^2 \delta(E_f - (E_i - \hbar\omega)) \]

where \( \omega \) is the energy of the photon. For an atomic transition, the long wavelength approximation is valid \( e^{-i k x} \rightarrow 1 \).

The momentum matrix element can be simplified by \( [x, H] = i \frac{\hbar}{m} p \),
\[ M_{ij} = -\frac{e}{mc} N_{ij} \langle 1 \mid s \cdot \epsilon_k^\lambda \rangle e^{-i k \cdot x} \langle 2 \mid p \rangle
\]
\[ = -\frac{e}{mc} N_{ij} \langle 1 \mid s \cdot \epsilon_k^\lambda \rangle \langle 2 \mid p \rangle
\]
\[ = \frac{e}{mc} N_{ij} i \frac{m}{\hbar} \langle 1 \mid [x, H] \rangle \langle 2 \mid p \rangle \cdot \epsilon_k^\lambda
\]
\[ = i \frac{e}{mc} N_{ij} \hbar \omega \langle 1 \mid s \cdot \epsilon_k^\lambda \rangle \langle 2 \mid p \rangle
\]

where \( \hbar \omega = E_i - E_f \) comes from operating with \( H_0 \) on the initial and final states. Putting together the components of the calculation

\[ \Gamma = \sum_{\lambda} \frac{1}{(2 \pi)^3} \int d^3 k \frac{2 \pi}{\hbar} \left( \frac{e \omega}{c} \right)^2 N_{ij}^2 |\langle 1 \mid s \cdot \epsilon_k^\lambda \rangle \langle 2 \mid p \rangle|^2 \delta(E_f - (E_i - \hbar \omega))
\]
\[ = \int d\Omega \frac{2 \pi^2}{2 \pi} \frac{\epsilon_i}{\hbar c} \sum_{\lambda} |\langle 100 \rangle |s \cdot \epsilon_k^\lambda |21 m\rangle|^2
\]
\[ = \int d\Omega \frac{\epsilon_i}{\hbar c} \sum_{\lambda} |\langle 100 \rangle |s \cdot \epsilon_k^\lambda |21 m\rangle|^2
\]
\[ = \int d\Omega \left( \frac{2 \pi^2}{2 \pi} \right) \frac{\epsilon_i}{\hbar c} \sum_{\lambda} |\langle 100 \rangle |s \cdot \epsilon_k^\lambda |21 m\rangle|^2
\]
\[ \Gamma = \frac{1}{3} \sum_m \Gamma_m
\]
\[ = \int d\Omega \frac{1}{3} \left( \frac{2 \pi^2}{2 \pi} \right) \frac{\epsilon_i}{\hbar c} \sum_{\lambda, m} |\langle 1 \mid r |21 \rangle|^2 \sum_m \sum_{\lambda, m} |\langle 00 \rangle |s \cdot \epsilon_k^\lambda |1 m\rangle|^2
\]
\[ = \int d\Omega \frac{1}{3} \left( \frac{2 \pi^2}{2 \pi} \right) \frac{\epsilon_i}{\hbar c} \sum_{\lambda, m} |\langle 1 \mid r |21 \rangle|^2 \sum_m \sum_{\lambda, m} |\langle 00 \rangle |s \cdot \epsilon_k^\lambda |1 m\rangle \langle 1 m |(s \cdot \epsilon_k^\lambda)^\dagger |00\rangle
\]
\[ = \int d\Omega \frac{1}{3} \left( \frac{2 \pi^2}{2 \pi} \right) \frac{\epsilon_i}{\hbar c} \sum_{\lambda, m} |\langle 1 \mid r |21 \rangle|^2 \sum_m \sum_{\lambda, m} |\langle 00 \rangle |s \cdot \epsilon_k^\lambda \rangle (l m |(s \cdot \epsilon_k^\lambda)^\dagger |00\rangle
\]
\[ = \int d\Omega \frac{1}{3} \left( \frac{2 \pi^2}{2 \pi} \right) \frac{\epsilon_i}{\hbar c} \sum_{\lambda, m} |\langle 1 \mid r |21 \rangle|^2 \sum_m \sum_{\lambda, m} |\langle 00 \rangle |(s \cdot \epsilon_k^\lambda)^\dagger |00\rangle
\]

where the sum over \( m \) can be expanded to a sum over all angular momentum states since the \( \hat{s} \) operator can only connect the \( |00\rangle \) state to \( l = 1 \) states. With a complete set of \( lm \)-states, \( 1 = \sum_{lm} |lm\rangle \langle lm| \) may be used. At this point the sum over polarization states can be moved inside the matrix element.

\[ \sum_{\lambda} \langle 00 \rangle \langle \hat{s} \cdot \epsilon_k^\lambda \rangle^2 |00\rangle = \langle 00 \rangle \sum_{\lambda=1,2} \langle \hat{s} \cdot \epsilon_k^\lambda \rangle^2 |00\rangle
\]

The two polarization vectors and the momentum direction define an orthonormal set of basis vectors for the position space. Adding in the third vector makes a complete projection operator,
\[ \Sigma \langle 00 | (\hat{x} \cdot \epsilon_{k\lambda}^*)^2 | 00 \rangle = \langle 00 | \sum_{\lambda=1,2,3} (\hat{x} \cdot \epsilon_{k\lambda}^*)^2 - (\hat{x} \cdot \epsilon_{k3}^*)^2 | 00 \rangle \]
\[ = \langle 00 | 1 - (\hat{x} \cdot \hat{k})^2 | 00 \rangle \]
\[ = \langle 00 | 1 - \cos^2(\theta) | 00 \rangle \]
\[ = \frac{2}{3} \]

Including this in the expression for \( \Gamma \), there is no longer any angular dependence so \( \int d\Omega = 4\pi \), and

\[ \Gamma = 4\pi \frac{1}{3} \frac{\omega^2}{2\pi c} \frac{e^2}{hc} \frac{2}{3} |\langle 10 | r | 21 \rangle|^2 \]
\[ = \frac{4}{9} \frac{\omega^2}{c} \frac{e^2}{hc} |\langle 10 | r | 21 \rangle|^2 \]

The last piece of work is to evaluate the radial matrix element. Using the radial wave functions in Appendix A,

\[ \langle 10 | r | 21 \rangle = \int dR R_{10} R_{21} = \frac{128}{81} \sqrt{\frac{2}{3}} a_0 \]

One way to proceed is to express \( a_0 = \frac{1}{\alpha m} \) and \( \omega = (1 - \frac{1}{2\pi}) \frac{a^2}{2} m = \frac{3}{8} \alpha^2 m \), where \( m = m_e = 0.511 \text{ MeV} \) is the electron mass and \( \frac{e^2}{hc} = \alpha = \frac{1}{137} \). Plugging in,

\[ \Gamma = \frac{4}{9} \left( \frac{3}{8} \right)^3 \frac{215}{37} \alpha^5 m_e = \frac{28}{37} \alpha^5 m_e \]

in units where \( \hbar = c = 1 \). Using \( \frac{1}{m_e} = 1.29 \times 10^{-21} \text{ sec} \), the lifetime is

\[ \tau = 1.6 \times 10^{-9} \text{ s} \]

### Detailed Balance

Consider the effects of placing the hydrogen atom of problem 5.40 in a blackbody radiation field of temperature \( T \), \( \langle N_\gamma(k, \lambda) \rangle = \frac{1}{e^{\omega/\pi} - 1} \) for each photon mode.

**a) How does the lifetime of the [2 p] state change?**

The lifetime changes in two ways. First, there is the possibility of excitation to higher energy states. This isn't what I had in mind, but it is there. What I had in mind was the increase in decay rate by stimulated emission. In the calculation of rates, the radiation matrix element must be treated in a more sophisticated fashion. Let the initial radiation state for a given mode be specified by \( |i_\gamma \rangle = \sum_n c_n |n \rangle \), where the sum is over different excitation states for that mode, weighted by coefficients for the different excitation levels. Then, focusing just on that single mode, the final state can be any excitation, so one must sum over the possibilities. The rate for emission, therefore, includes a factor of

\[ \Gamma \sim \sum_n |\langle n' | a^+ | i_\gamma \rangle|^2 \]
\[ = \sum_n \langle i_\gamma | a | n' \rangle \langle n' | a^+ | i_\gamma \rangle \]
\[ = \langle i_\gamma | aa^+ | i_\gamma \rangle \]
\[ = \langle i_\gamma | N_\gamma + 1 | i_\gamma \rangle \]
\[ = \langle N_\gamma \rangle + 1 \]
\[ = \frac{1}{e^{\omega/\pi} - 1} + 1 \]
\[ = \frac{e^{\omega/\pi}}{e^{\omega/\pi} - 1} \]
In the limit of large $T$, the rate is increased by $\frac{\omega}{\omega_0}$. Note that one does not need explicit values for $c_n$, just the expectation value of $N_γ$.

b) Find the relative probability for the atom to be in the $|1 \ s\rangle$ and $|2 \ p\rangle$ states.

In equilibrium, the total rate for transitions from $|1 \ s\rangle \rightarrow |2 \ p\rangle$ must be balanced by the rate for $|2 \ p\rangle \rightarrow |1 \ s\rangle$. The total rate from one state to another is the rate per atomic state times the probability to be in a particular state $P_i$. Thus, equilibrium requires

$$P_{1s} \Gamma_{1s \rightarrow 2p} = P_{2p} \Gamma_{2p \rightarrow 1s}$$

or

$$\frac{P_{2p}}{P_{1s}} = \frac{\Gamma_{1s \rightarrow 2p}}{\Gamma_{2p \rightarrow 1s}}$$

The rates involve the same phase space integrals and atomic matrix elements, but have different matrix elements for the electromagnetic field. In addition, one must be careful to properly sum over atomic polarization states. In general, one presumes that in thermal equilibrium, the atom is not polarized and so one averages over initial polarizations. The rate must include all possible final polarization states, so one sums over final atomic polarizations.

$$\frac{P_{2p}}{P_{1s}} = \frac{\Gamma_{1s \rightarrow 2p}}{\Gamma_{2p \rightarrow 1s}}$$

$$= \frac{\sum_{m_l} |(2p, m_l) \cdot |1\ s\rangle|^2 \langle N_{\gamma}|}{\sum_{m_l} |(2p, m_l) \cdot |1\ s\rangle|^2 \langle N_{\gamma}+1|}$$

$$= \frac{g_{2p} \langle N_{\gamma}|}{g_{1s} \langle N_{\gamma}+1|}$$

$$= 3 \ e^{-\omega_0/T}$$

The factor $g_{2p} = (2l + 1) = 3$ accounts for three degenerate $m_l$ states in the $2 \ p$ level. There is only a single $1 \ s$ state, so $g_{1s} = 1$. The Boltzman factor of $e^{-\omega_0/T}$ arises from the thermal statistics of the photon field.

### Coulomb cross-section

Using the Born approximation show that the Coulomb differential cross-section, i.e. the cross-section for scattering an incident charged particle with charge $z_1$ from a Coulomb potential with charge $z_2$, is

$$\frac{d\sigma}{d\Omega} = \frac{e^4 z_1^2 z_2^2}{16 E^2 \sin^4(\frac{\theta}{2})}$$

where $E$ is the incident energy, and $\theta$ is the scattering angle in the lab frame.

### solution

The problem and solution are given in units where $\hbar = c = 1$.

This problem can be treated as a problem in time dependent perturbation theory, by calculating the transition rate from plane waves with initial momentum $k_i$ to plane waves of final momentum $k_f$. The cross-section is given by $\sigma = \Gamma / \phi$. Here, $\phi$ is the flux, which for plane wave states includes the velocity $v$, and $\Gamma$ is the total transition rate.
\[
\Gamma = \frac{1}{(2\pi)^3} \int d^3 k_f \Gamma_{k\rightarrow k_f} \\
= \frac{1}{(2\pi)^2} \int k_f^2 \, dk_f \, d\Omega \, \Gamma_{k\rightarrow k_f}
\]

The normalization of the initial momentum state appears in both the flux and in \(\Gamma\), and so drops out of the calculation.

The differential cross-section is derived from the differential transition rate per unit solid angle

\[
\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} \int k_f^2 \, dk_f \, \Gamma_{k\rightarrow k_f}
\]

The transition rate to a given direction is found from Fermi's golden rule

\[
\Gamma_{k\rightarrow k_f} = 2\pi |\langle k_f | V | k_i \rangle|^2 \delta(E_f - E_i)
\]

First, consider the matrix element, given for the momentum transfer \(q = k_f - k_i\)

\[
\langle k_f | V | k_i \rangle = \int d^3 x \, e^{-i k_f \cdot x} \frac{\mathbf{k}_{iqr}^2}{\mathbf{m}} \, e^{i \mathbf{k} \cdot x}
\]

\[
= \int r^2 \, dr \, d(\cos \theta) \, d\varphi \, e^{-i q \cdot x} \frac{\mathbf{k}_{iqr}^2}{\mathbf{m}}
\]

\[
= 2\pi e^2 z_1 z_2 \int r \, dr \, d(\cos \theta) \, r \, e^{-i q \cos(\theta)}
\]

This integral is not well defined, but it can be controlled by adding a term \(e^{-\mu r}\) (equivalent to giving the photon a small mass) which makes the integral convergent, and then taking the limit \(\mu \to 0\).

\[
e^{-\mu r} \int dr \, (e^{i q r} - e^{-i q r}) = \int dr \, (e^{(-\mu+i q) r} - e^{(-\mu-i q) r})
\]

\[
= \left[ \frac{e^{(-\mu+i q) r}}{-\mu+i q} - \frac{e^{(-\mu-i q) r}}{-\mu-i q} \right]_0^\infty
\]

\[
= \frac{2iq}{\mu^2+q^2}
\]

Taking \(\mu \to 0\) limit, the squared matrix element is

\[
|\langle k_f | V | k_i \rangle|^2 = \frac{16\pi^2}{q^2} \, e^4 z_1^2 z_2^2
\]

Next consider the \(\delta\)-function, using \(k^2 = 2mE\), so that \(\int k^2 \, dk = \sqrt{2mE} \, m \, dE\). Putting it all together

\[
\frac{d\sigma}{d\Omega} = \frac{1}{\nu} \frac{d\Gamma}{d\Omega} = \frac{1}{\nu} \frac{1}{(2\pi)^2} \int \sqrt{2mE_f} \, m \, dE_f \, 2\pi \frac{16\pi^2}{q^2} \, e^4 z_1^2 z_2^2 \delta(E_f - E_i)
\]

\[
= \frac{1}{\nu} \frac{1}{(2\pi)^2} \sqrt{2mE} \, m \, 2\pi \frac{16\pi^2}{q^2} \, e^4 z_1^2 z_2^2 \delta(E_f - E_i)
\]

\[
= 4 \frac{\epsilon m}{\nu} \frac{e^4 z_1^2 z_2^2}{q^4}
\]

\[
= 4 \frac{e^4 z_1^2 z_2^2}{16m^2 v^2 \sin^2(\frac{q}{2})}
\]

\[
= \frac{e^4 z_1^2 z_2^2}{16E^2 \sin^2(\frac{q}{2})}
\]

where \(q^2 = (k_f - k_i)^2 = 2k^2(1 - \cos \theta) = 4m^2 v^2 \sin^2(\frac{\theta}{2}) = 8mE \sin^2(\frac{\theta}{2})\)