2. Number of galaxies in the universe

The HUDF in figure 29.5 contains \(\sim 10,000\) galaxies within a patch of sky of 3x3 square arcmin.

(a) What is the solid angle \(\Omega_{\text{HUDF}}\) of this patch, in ster?

1 arcmin = \(\pi/(60 \times 180) = 2.9 \times 10^{-4}\) radian. So

\[
\Omega_{\text{HUDF}} = 9 \text{arcmin}^2 = 7.6 \times 10^{-7} \text{ ster}.
\]

(b) What fraction \(f\) of the entire sky does this represent?

\[
f = \frac{\Omega_{\text{HUDF}}}{4\pi} = 6.1 \times 10^{-8}.
\]

(c) Use this to estimate the total number of galaxies \((N_{\text{gal}})\) that would be observed from similar patches over the full sky.

\[
N_{\text{gal}} \approx 10^4/f \approx 1.6 \times 10^{11}.
\]

(d) Assuming the HUDF detects all galaxies out to a distance of \(\sim 10\) Gly, estimate the average number density \((n_{\text{gal}})\) of galaxies in the universe, in Mly\(^{-3}\).

Volume of sphere to 10 Gly is \(V = (4\pi/3) \times 10^{12}\) Mly\(^3\). So

\[
n_{\text{gal}} \approx \frac{N_{\text{gal}}}{V} \approx 0.038 \text{ Mly}^{-3}.
\]

(e) What does this imply for the average separation distance \(d_{\text{gal}}\) (in Mly) between galaxies? How does this compare to the distance \(d_{\text{And}}\) between the MW and Andromeda?

\[
d_{\text{gal}} \approx n_{\text{gal}}^{-1/3} \approx 3 \text{ Mly} \gtrsim d_{\text{And}} \approx 2 \text{ Mly}.
\]

(f) Discuss potential reasons for any differences between \(d_{\text{And}}\) and \(d_{\text{gal}}\).

Andromeda and Milky Way are part of a small cluster known as the “Local Group”, so should be slightly denser than average over universe. But the two values are actually remarkably close.
1. **Critical universe redshift.**

Consider a critical universe $\Omega_m = 1$ without dark energy ($\Omega_\Lambda = 0$) and a local Hubble constant equal to the currently inferred best value $H_0 \approx 70 \text{ (km/s)/Mpc}$.

(a) Derive a formula for redshift $z$ vs. distance $d$ (in Mpc).

Writing past time as $t = -H_0d/c$, we find for critical universe without dark energy,

$$z(d) = \frac{1}{R(t)} - 1 = \left(1 - \frac{3H_0d}{2c}\right)^{-2/3} - 1 \Rightarrow$$

$$z(d) \approx \left(1 - \frac{d}{2860 \text{ Mpc}}\right)^{-2/3} - 1 \quad (30.1)$$

(b) Show that for small distances $d \ll c/H_0$, this recovers the simple linear Hubble law $cz = H_0d$.

$$z(d) \approx 1 - (-2/3)(3H_0d)/(2c) - 1 \Rightarrow \boxed{zc = H_0d} \quad (30.2)$$

(c) Compute the time since the Big Bang, in Gyr.

$$R(t) = (1 + (3H_0t)/2)^{2/3} = 0 \Rightarrow -t = 2/(3H_0) = \boxed{9.5 \text{ Gyr}} \quad (30.3)$$

(d) Compare this time to the age of a Globular cluster with a main-sequence turnoff at luminosity $L_{10} = 0.75L_\odot$.

$$t_{10} = 10 \text{ Gyr} \left(\frac{L}{L_\odot}\right)^{-2/3} = 10 \text{ Gyr} (0.75)^{-2/3} = \boxed{12.1 \text{ Gyr}} \quad (30.4)$$

(e) What does this say about the viability of this as a model for our universe? What about closed-universe models with $\Omega_m > 1$? (Assume the above Hubble constant measurement is accurate, and that there is no dark energy.)

Since $t_{10} = 12.1 \text{ Gyr} > 9.6 \text{ Gyr} = t_{age}$, this is NOT a viable model, since the oldest stars are older than this estimated age of the universe. Moreover, all models $\Omega_m > 1$ would be even younger, and so they are also not viable.
2. *Empty Universe:*

Next consider the case of an effectively “empty” universe with $\Omega_m = \Omega_\Lambda = 0$, that is again expanding with a locally measured Hubble constant $H_o \approx 70 \text{ (km/s)/Mpc}$. Repeat parts a-d of previous exercise for this case of an empty universe. What does the result in part (d) here say about the formal viability of this as a model for our universe?

a. Writing past time as $t = -H_o d/c$, we find for *empty* universe without dark energy,

$$z(d) = \frac{1}{R(t)} - 1 = \left(1 - \frac{H_o d}{c}\right)^{-1} - 1 \Rightarrow$$

$$z(d) \approx \frac{d}{4920 \text{ Mpc}} - 1 \quad (30.5)$$

b.

$$z(d) \approx 1 + H_o d/c - 1 \Rightarrow z c = H_o d \quad (30.6)$$

c.

$$R(t) = (1 + H_o t) = 0 \Rightarrow -t = \frac{14.3 \text{ Gyr}}{H_o} \quad (30.7)$$

d., e. Since now $t_{to} = 12.1 \text{ Gyr} < 14.3 \text{ Gyr} = t_{age}$, this *IS* a viable model, since the oldest stars are still younger than this estimated age of the universe. Also, though models $\Omega_m > 1$ would be younger, some might still be viable.
4. Ex. 30.3

Empty vs. critical universe (Ex. 30.3)

(a) For the above empty universe model, invert the formula for \( z(d) \) to derive an expression for distance as a function of redshift \( z \). For this use the notation \( d_0(z) \), where the subscript “0” denotes the null value of \( \Omega_m \).

\[
z(d_0) = \frac{1}{1 - H_0 d_0/c} - 1 \Rightarrow d(z_0) = \frac{z_0}{1 + z_0} \frac{c}{H_0} = \frac{z_0}{1 + z_0} 14.5 \text{ Gyr}
\]

(b) If a distance measurement is accurate to 10%, at what minimum redshift \( z_0 \) can one observationally distinguish the redshift vs. distance of an empty universe from a strictly linear Hubble law \( d = cz/H_0 \).

**Linear Hubble has** \( d(z) = cz/H_0 \), whereas **empty universe has** \( d_0 (z/(1 + z)) c/H_0 \). So with 10% reduction in linear Hubble, we have

\[
0.9 \frac{z_0 c}{H_0} = \frac{z_0}{1 + z_0} c/H_0 \Rightarrow z_0 = 0.11
\]

(c) Using the above results from part a, now derive an analogous distance vs. redshift formula \( d_1(z) \) for the critical universe with \( \Omega_m = 1 \) (and \( \Omega_\Lambda = 0 \)).

\[
d_1(z) = \left[1 - \frac{1}{(1 + z)^{3/2}}\right] \frac{2c}{3H_0} = \left[1 - \frac{1}{(1 + z)^{3/2}}\right] 14.5 \text{ Gyr}
\]

(d) Again if a distance measurement is accurate to 10%, at what minimum redshift \( z_1 \) can one observationally distinguish the redshift vs. distance of such a critical universe from a strictly linear Hubble law.

\[
0.9 z_1 = (2/3) \left[1 - \frac{1}{(1 + z_1)^{3/2}}\right] \Rightarrow z_1 = 0.088
\]

(e) Finally, again with a distance measurement accurate to 10%, at what minimum redshift \( z_{10} \) can one observationally distinguish the redshift vs. distance of a critical universe from an empty universe?

\[
0.9 \frac{z_{10}}{1 + z_{10}} = (2/3) \left[1 - \frac{1}{(1 + z_{10})^{3/2}}\right] \Rightarrow z_{10} = 0.59
\]
4. Inverted pendulum analog for dark energy

Consider an inverted pendulum in which a mass $m$ is balanced vertically on a rigid but massless rod of length $\ell$ in the presence of a downward gravity $-g$.

(a) For small angle displacements $\theta \ll 1$, show that $\ddot{\theta} = +(g/\ell)\theta$.

Taking $\theta$ to be the angle of the rod away from purely vertical, the gravitational force on the mass, $mg$, exerts a torque $mg\ell \sin \theta$ on the rod, whose moment of inertia is $I = m\ell^2$. The rotational equation of motion is then $I\ddot{\theta} = mg\ell \sin \theta$. But for $\theta \ll 1$, we have $\sin \theta \approx \theta$, and so using $I = m\ell^2$, we find $\ddot{\theta} = +(g/\ell)\theta$.

(b) How does this differ from the usual case of a pendulum suspended downward?

For downward pendulum, $\ddot{\theta} = -(g/\ell)\theta$.

(c) For the inverted case, obtain solutions for $\theta(t)$ given initial conditions $\theta(t = 0) = 0$ and $\dot{\theta}(t = 0) = \dot{\theta}_o$.

Defining $\omega = \sqrt{g/\ell}$, the general solution with constants $A$ and $B$ is $\theta(t) = A \sinh(\omega t) + B \cosh(\omega t)$. Applying conditions, we find $B = 0$ and $A = \dot{\theta}_o / \sqrt{g/\ell}$, giving

$$\theta(t) = \dot{\theta}_o \sqrt{\ell/g} \sinh(\sqrt{g/\ell} t)$$ (31.1)

(d) Similarly for the matterless Dark Energy model of equation (31.1) with $\Omega_m = 0$, obtain solutions for $R(t)$ given $R(t = 0) = 0$ and $\dot{R}(t = 0) = H_o$.

For $\Omega_m = 0$, eqn. (31.1) becomes $\ddot{R} = \Omega_\Lambda H_o^2 R$. With the stated initial conditions, this has the solution

$$R(t) = \left(1/\sqrt{\Omega_\Lambda}\right) \sinh(\sqrt{\Omega_\Lambda} H_o t)$$ (31.2)

(e) What then are the inverted pendulum analogs to $R$, $H_o$, and $\Omega_\Lambda$?

$R \leftrightarrow \theta$, $\Omega_\Lambda \leftrightarrow g/(\ell \dot{\theta}_o^2)$, $H_o \leftrightarrow \dot{\theta}_o$